Logarithms and the Harmonic Series

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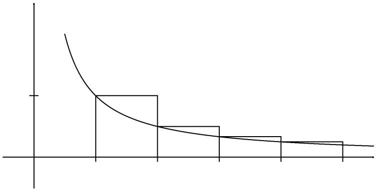
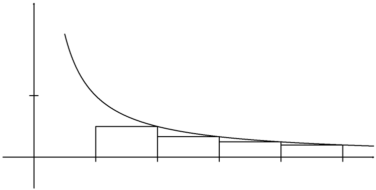
Logarithms

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots \tag{2.27}$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \tag{2.28}$$

Harmonic series and logarithms





Harmonic series and logarithms

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

$$\frac{1}{n} \leq \int_{n-1}^{n} \frac{1}{t}\,\mathrm{d}t \leq \frac{1}{n-1}$$

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{1}{t} dt \leq \sum_{k=1}^{n} \frac{1}{k-1}$$

$$H_n - 1 \leq \ln n \leq H_{n-1}$$

$$\ln(n+1) < H_n < 1 + \ln n$$

Integral test

f is monotone and decreasing:

$$\int_N^{\infty} f(t)\,\mathrm{d}t \le \sum_{n=N}^{\infty} f(n) \le f(N) + \int_N^{\infty} f(t)\,\mathrm{d}t$$

$$\int_N^{K+1} f(t)\,\mathrm{d}t \le \sum_{n=N}^{K} f(n) \le f(N) + \int_N^{K} f(t)\,\mathrm{d}t$$

Euler's constant

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) = \lim_{n\to\infty}(H_{n-1} - \ln n)$$

$$\gamma \approx 0{\cdot}577{,}215{,}665$$

$$x_n = H_{n-1} - \ln n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln n$$

$$y_n = H_n - \ln n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \ln n = x_n + \frac{1}{n}$$

$$x_n < x_{n+1} \qquad y_n > y_{n+1}$$

Inequalities with logarithms

$$\ln x \le x - 1$$
$$\ln \frac{1}{x} \le \frac{1}{x} - 1$$
$$-\ln x \le \frac{1}{x} - 1$$
$$\ln x \ge 1 - \frac{1}{x}$$

$$1 - \frac{1}{x} \le \ln x \le x - 1$$
$$1 - \frac{1}{x+1} \le \ln(1+x) \le x$$

Inequalities with logarithms

$$\frac{1}{n+1} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$$

$$x_{n+1} - x_n = \frac{1}{n} - \ln(n+1) + \ln n = \frac{1}{n} - \ln\left(1+\frac{1}{n}\right) > 0$$

$$y_n - y_{n+1} = \ln(n+1) - \ln n - \frac{1}{n+1} = \ln\left(1+\frac{1}{n}\right) - \frac{1}{n+1} > 0$$

Approximating H_n

$H_n \approx \ln n - \gamma \approx 10 \Rightarrow n \approx e^{10-\gamma} \approx 123{,}669{\cdot}68$
$H_{12366} \approx 9{\cdot}999{,}962$
$H_{12367} \approx 10{\cdot}000{,}043$

$$H_{n-1} - \ln n < \gamma < H_n - \ln n$$
$$H_{n-1} < \gamma + \ln n < H_n$$
$$\gamma + \ln(n-1) < H_{n-1} < \gamma + \ln n < H_n$$
$$9{\cdot}999{,}921{,}7 < H_{12366} < 10{\cdot}000{,}025{,}8 < H_{12367}$$

Approximating H_n

$$
\begin{aligned}
1 &= \ln 2 - \ln 1 = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots \\
\frac{1}{2} &= \ln 3 - \ln 2 + \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} - \frac{1}{5 \cdot 2^5} + \dots \\
\frac{1}{3} &= \ln 4 - \ln 3 + \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} - \frac{1}{5 \cdot 3^5} + \dots \\
\frac{1}{4} &= \ln 5 - \ln 4 + \frac{1}{2 \cdot 4^2} - \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} - \frac{1}{5 \cdot 4^5} + \dots \\
&\vdots \\
\frac{1}{n-1} &= \ln n - \ln(n-1) + \frac{1}{2 \cdot (n-1)^2} - \frac{1}{3 \cdot (n-1)^3} + \dots
\end{aligned}
$$

Approximating H_n

$$
\begin{aligned}
H_{n-1} =& 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \\
=& \ln n \\
& + \frac{1}{2}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2}\right) \\
& - \frac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{(n-1)^3}\right) \\
& + \frac{1}{4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{(n-1)^4}\right) \\
& - \frac{1}{5}\left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \cdots + \frac{1}{(n-1)^5}\right) + \ldots
\end{aligned}
$$

Approximating H_n

$$\begin{aligned}
\gamma &= \lim_{n\to\infty}(H_{n-1} - \ln n) \\
&= \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2} - \frac{1}{3}\sum_{m=1}^{\infty}\frac{1}{m^3} + \frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{m^4} - \frac{1}{5}\sum_{m=1}^{\infty}\frac{1}{m^5} + \dots \\
&= \frac{\zeta(2)}{2} - \frac{\zeta(3)}{3} + \frac{\zeta(4)}{4} - \frac{\zeta(5)}{5} + \dots
\end{aligned}$$

$$\begin{aligned}
H_{n-1} - \ln n - \gamma &= -\frac{1}{2}\sum_{m=n}^{\infty}\frac{1}{m^2} + \frac{1}{3}\sum_{m=1}^{\infty}\frac{1}{m^3} - \dots \\
H_{n-1} &< \ln n + \gamma - \frac{1}{2}\sum_{m=n}^{\infty}\frac{1}{m^2} + \frac{1}{3}\sum_{m=1}^{\infty}\frac{1}{m^3}
\end{aligned}$$

Approximating H_n

$$H_{n-1} < \ln n + \gamma - \frac{1}{2}\sum_{m=n}^{\infty}\frac{1}{m^2} + \frac{1}{3}\sum_{m=1}^{\infty}\frac{1}{m^3}$$

$$\sum_{m=n}^{\infty}\frac{1}{m^2} > \int_n^{\infty}\frac{\mathrm{d}t}{t^2} = \frac{1}{n}$$

$$\sum_{m=n}^{\infty}\frac{1}{m^3} < \frac{1}{n^3} + \int_n^{\infty}\frac{\mathrm{d}t}{t^3} = \frac{1}{n^3} - \frac{2}{n^2} = \frac{2+n}{2n^3}$$

$$H_{12366} < \ln 12367 + \gamma - \tfrac{1}{2\cdot 12367} + \tfrac{12369}{2\cdot 12369^2} < 9{\cdot}999{,}962{,}2$$

Problem 2.4.7

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = \frac{1}{2}\log 2$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$$

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots = \frac{\pi}{4} + \frac{\ln 2}{2}$$

Estimate with the first 1000 terms:

$$\frac{\pi}{4} + \frac{\ln 2}{2} = 1{\cdot}131{,}971{,}753{,}677{,}421$$
$$S_{1000} = 1{\cdot}130{,}972{,}254{,}176{,}419$$

Problem 2.4.10

Kempner series

$$
\begin{aligned}
1+\frac{1}{2}+\cdots+\frac{1}{8} &\leq 8 \\
\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{88} &\leq \frac{8\cdot 9}{10} \\
\frac{1}{100}+\frac{1}{101}+\cdots+\frac{1}{888} &\leq \frac{8\cdot 9^2}{10^2} \\
&\vdots
\end{aligned}
$$

$$S \leq 8\left(1+\frac{9}{10}+\frac{9^2}{10^2}+\dots\right)=\frac{8}{1-\frac{9}{10}}=80$$

Kempner series

Neither 8 nor 9

$$7 + \frac{7 \cdot 8}{10} + \frac{7 \cdot 8^2}{10^2} + \cdots = \frac{7}{1 - \frac{8}{10}} = 35$$

No digit 1:

$$\frac{8}{2} + \frac{8 \cdot 9}{2 \cdot 10} + \frac{8 \cdot 9^2}{2 \cdot 10^2} + \cdots = 40$$

Problem 2.4.13

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

$$\frac{1}{\sqrt{k+1}} \leq \frac{2}{\sqrt{k+1} + \sqrt{k}} \leq \frac{1}{\sqrt{k}}$$

$$\frac{1}{\sqrt{k+1}} \leq 2(\sqrt{k+1} - \sqrt{k}) \leq \frac{1}{\sqrt{k}}$$

$$2(\sqrt{k+1} - \sqrt{k}) \leq \frac{1}{\sqrt{k}} \leq 2(\sqrt{k+1} - \sqrt{k-1})$$

$$2\sqrt{n+1} - 2 \leq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$$

Problem 2.4.13

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

n	S_n	$2\sqrt{n+1}-2$	$2\sqrt{n}$
10	5·020,998	4·633	6·325
100	18·589,604	18·100	20
1,000	61·801,009	61·277	63·246
10,000	198·544,645	198·010	200
100,000	630·996,759	630·459	632·455
1,000,000	1,998·540,145	1,998·001	2,000
10,000,000	6,323·095,123,940	6,322·556	6,324·555

Problem 2.4.14

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \ln(2) + O\left(\frac{1}{n}\right)$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} = \ln(2n) + \gamma$$

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} = \ln 2 + \frac{\gamma}{2} + \frac{1}{2}\ln n + O\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2}\ln n\right) = \ln 2 + \frac{\gamma}{2}$$

Problem 2.4.15

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2nr-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2ns}\right)$$

$$\begin{aligned} S &\approx \left(\ln 2 + \frac{\gamma}{2} + \frac{\ln(nr)}{2}\right) - \frac{1}{2}(\gamma + \ln(ns)) \\ &= \ln 2 + \frac{1}{2}\ln(nr) - \ln(ns) = \ln 2 + \frac{1}{2}\ln\frac{r}{s} \end{aligned}$$

Ant on a rubber rope

$$\frac{1}{2000}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

$$\ln n+\gamma \approx 2000$$
$$n \approx e^{2000-\gamma}$$

Ant on a rubber rope - example with small length

Table shows: position of ant, length, ratio

	0	2	0
A	1	2	$\frac{1}{2}$
H	$\frac{3}{2}$	3	$\frac{1}{2}$
A	$\frac{5}{2}$	3	$\frac{5}{6}$
H	$\frac{10}{3}$	4	$\frac{5}{6}$
A	$\frac{13}{3}$	4	$\frac{13}{12}$

Ant on a rubber rope - discrete case

$$\theta(n-1) = \frac{\alpha}{c+v} + \frac{\alpha}{c+2v} + \cdots + \frac{\alpha}{c+(n-1)v}$$

$$\theta(n-1) \geq \int_1^n \frac{\alpha}{c+vt}\,\mathrm{d}t = \left[\frac{\alpha}{v}\ln(c+vt)\right]_1^n$$

$$\theta(n-1) \geq \frac{\alpha}{v}\ln\left(\frac{c+vn}{c+v}\right)$$

Ant on a rubber rope - continuous case

$$y'(t) = \alpha + \frac{vy(t)}{c + vt}$$

$$\phi(t) = \frac{y(t)}{c + vt}$$

$$\begin{aligned}
y(t) &= \phi(t)(c + vt) \\
y'(t) &= \phi'(t)(c + vt) + v\phi(t) \\
y'(t) &= \phi'(t)(c + vt) + \frac{vy(t)}{c + vt} \\
y'(t) - \frac{vy(t)}{c + vt} &= \phi'(t)(c + vt) \\
\alpha &= \phi'(t)(c + vt) \\
\frac{\alpha}{c + vt} &= \phi'(t)
\end{aligned}$$

Ant on a rubber rope - continuous case

$$\phi'(t) = \frac{\alpha}{c + vt}$$

$$\phi(t) = \int_0^t \frac{\alpha}{c + vx}\,\mathrm{d}x$$

$$\phi(t) = \frac{\alpha}{v} \ln\left(\frac{c + vt}{c}\right)$$

$$\frac{\alpha}{v} \ln\left(\frac{c + vT}{c}\right) = 1 \qquad \Leftrightarrow \qquad T = \frac{c}{v}\left(e^{v/\alpha} - 1\right)$$