van Rooij, Schikhof: A Second Course on Real Functions

Notes from [vRS].

Introduction

A monotone function is Riemann integrable.
A continuous function is Riemann integrable.
A differentiable function is continuous.
A continuous function has an antiderivative.

Theorem. *Every function that has an antiderivative is Darboux continuous.*

$\mathcal{M}on$	the set of monotone functions
$\mathcal{D}$	the set of differentiable functions
$\mathcal{C}$	the set of continuous functions
$\mathcal{R}$	the set of Riemann integrable functions
$\mathcal{D}'$	the set of derivative functions (functions with antiderivatives)
$\mathcal{DC}$	the set of Darboux continuous functions

1 Monotone functions

1.1 Continuity of monotone functions

1.2 Indefinite integrals of monotone functions (convex functions)

TODO Φ_1, Φ_2

Theorem. *(2.2) Let $I \subset \mathbb{R}$ be an interval and let $f\colon I \to \mathbb{R}$. The following conditions are equivalent.*
(α) f is convex.
(β) $\Phi_1 f$ is an increasing function in each variable.
(γ) $\Phi_2 f \geq 0$.

$$D_r f(x) := \lim_{y\to x^+} \frac{f(y)-f(x)}{y-x}$$
$$D_l f(x) := \lim_{y\to x^-} \frac{f(y)-f(x)}{y-x}$$

1.3 Differences of monotone functions

Let $P\colon a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $\langle a, b\rangle$. Then we define for a function $f\colon \langle a, b\rangle \to \mathbb{R}$

$$L_P(f) := \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

The *length of the graph* of f is by definition

$$L(f) := \sup\{L_p(f)\colon P \text{ is a partition of } \langle a, b\rangle\}$$

($(L(f))$ may be ∞.)

total variation of f

$$\operatorname{Var} f := \sup\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|; a = x_0 < \cdots < x_n = n \text{ is a partition of } \langle a, b\rangle\}$$

Definition. (3.1) TODO bounded variation

First, one defines the integral of a step function in the obvious way. Then by continuity one defines the integral for functions that are limits of uniformly convergent sequences of step functions. These functions are said to be *Cauchy-Bourbaki integrable.*

Theorem. *(3.3) The following conditions on $f\colon \langle a, b\rangle \to \mathbb{R}$ are equivalent.*
(α) f is the uniform limit of a sequence of step functions.
(β) f is the uniform limit of a sequence of functions of bounded variation.
(γ) For each $x \in \langle a, b\rangle$, $f(x+)$ and $f(x-)$ exist. (It follows that an f satisfying (γ) is bounded.)

1.4 Differentiability of monotone functions

Definition. (4.1) Let $E \subseteq \mathbb{R}$. We call E a *null set* (or a *negligible set*) if for every $\varepsilon > 0$ there exist intervals $I_1, I_2, \dots$ covering E and such that the sum of their lengths is at most ε. To avoid laborious circumlocutions, we call $\sum_{i=1}^{\infty} L(I_i)$ the total length of the intervals $I_1, I_2, \dots,$ without requiring these intervals to be pairwise disjoint. (The length of an interval I is denoted by $L(I)$.)

Theorem. *(4.3) Let $E \subset \mathbb{R}$ be a null set. Then there exists a continuous increasing function on U that is differentiable at no point of E.*

Theorem (Lebesgue)**.** *(4.10) A monotone function $f\colon \langle a, b\rangle \to \mathbb{R}$ is differentiable almost everywhere on $\langle a, b\rangle$.*

2 Subsets of $\mathbb{R}$

2.5 Small sets

Theorem (Baire)**.** *(5.4)* $\mathbb{R}$ *is not meagre.*

Theorem. *(5.5) There exist nowhere dense sets that are not null. There exist null sets that are not meagre. More than that* $\mathbb{R}$ *can be written as the union of a null set and a meagre set.*

The example given here is constructed as a complement of $\bigcap U_k$, where $U_k := (r_i - 2^{-ki}, r_i + 2^{-ki})$ and $\mathbb{Q} = \{r_i; i = 1, 2, \dots\}$ is an enumeration of rational numbers.[1]

2.6 F_σ-sets and G_δ-sets

Theorem. *(6.2)*

(i) Every closed set is an F_σ*. A union of countably many* F_σ*-sets is an* F_σ*. If* A_1, A_2 *are* F_σ*-sets, then so is* $A_1 \cap A_2$*.*

(ii) Every open set is a G_δ*. An intersection of countably many* G_δ*-sets is a* G_δ*. If* B_l*,* B_2 *are* G_δ*-sets, then so is* $B_1 \cup B_@$*.*

(iii) A set is an F_σ *if and only if its complement is a* G_δ*.*

(iv) An interval is both an F_σ *and a* G_δ*. Every open set is an* F_σ*; every closed set is a* G_δ*.*

(v) Every countable set is an F_σ*.*

Theorem. *(6.3) A subset of* $\mathbb{R}$ *is meagre if and only if its complement contains a* G_δ*-set that is dense in* $\mathbb{R}$*.*

$\mathbb{Q}$ is not a G_δ. (Otherwise $\mathbb{R} \setminus \mathbb{Q}$ would be meagre and, consequently, $\mathbb{R}$ would be meagre.)

2.7 Behaviour of arbitrary functions

2.7.1 The set of maxima and minima of an arbitrary function

Let $f\colon \mathbb{R} \to \mathbb{R}$. We say that s is a *local maximum* of f if there exist an $x \in \mathbb{R}$ and a $\delta > 0$ such that $f(x) = s$ while $f(y) \le s$ for $x - \delta < y < x + \delta$. Similarly one can define *local minimum.* If s is a local maximum or a local minimum we say that s is a *local extremum* of f

Theorem. *(7.2) Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be any function. Then*

$$\{a \in \mathbb{R}; a \textit{ is a local extremum of } f\}$$

is a countable set.

[1] Fat Cantor sets (also called SmithVolterraCantor sets) are other examples of such sets. `http://en.wikipedia.org/wiki/Fat_Cantor_set`

2.7.2 The set of continuity points of an arbitrary function

$$C_f = \{x \in \mathbb{R}; f \text{ is continuous at } x\}$$

Theorem. *(7.5) Let $f\colon \mathbb{R} \to \mathbb{R}$. Then the set of points of continuity of f is a G_δ.*

Corollary. *(7.6) There is no function $\mathbb{R} \to \mathbb{R}$ that is continuous at each rational point and discontinuous at each irrational point.*

Theorem. *(7.7) Let f be any function on $\mathbb{R}$. Then there exist only countably many points x of $\mathbb{R}$ for which f is not continuous at x but $f(x+)$ exists.*

Corollary. *(7.8) If f is any function on $\mathbb{R}$ then the set*

$$\{x \in \mathbb{R}; \lim_{y \to x} f(y) = \infty\}$$

is countable.

Corollary. *(7.9) A left continuous function on $\mathbb{R}$ has only countably many discontinuities.*

Theorem. *(7.11) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the set of points where it is differentiable is a countable intersection of F_σ-sets.*

Lemma. *(7.12) There exist subsets of $\mathbb{R}$ that are not $F_{\sigma\delta}$-sets.*

Theorem. *(7.14) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Then the set $\{a \in \mathbb{R}; f$ is not increasing at $a\}$ is a G_δ without isolated points.*

Corollary. *(7.15) There is no continuous function $f\colon \mathbb{R} \to \mathbb{R}$ for which $\{a \in \mathbb{R}; f$ is not increasing at $a\} = \mathbb{R} \setminus \mathbb{Q}$.*

[2]

2.7.3 The set of points where a continuous function is differentiable

2.7.4 The set of points where a continuous function is increasing

3 Continuity

3.8 Continuous functions

Theorem. *(8.1) Let I be an interval.*

(i) If $f, g \in \mathcal{C}$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathcal{C}$ and $fg \in \mathcal{C}$. ($\mathcal{C}$ is an algebra.)

(ii) If $f, g \in \mathcal{C}$, then $f \vee g \in \mathcal{C}$ and $f \wedge g \in \mathcal{C}$. ($\mathcal{C}$ is a lattice.)

(iii) If $f_1, f_2, \dots \in \mathcal{C}$ and $f = \lim_{n\to\infty} f_n$ uniformly, then $f \in \mathcal{C}$. ($\mathcal{C}$ is uniformly closed.)

(iv) If $f \in \mathcal{C}$ and $f(x) \neq 0$ for all $x \in I$, then $1/f \in \mathcal{C}$.

[2] TODO Example 7.16

(v) *If $f \in \mathcal{C}$ and $g\colon f(I) \to \mathbb{R}$ is continuous, then $g \circ f \in \mathcal{C}$.*

(vi) *If $f \in \mathcal{C}$, then $f(I)$ is an interval or, if f is constant, a singleton. Moreover, if f is injective, then f is strictly monotone and $f^{-1}\colon f(I) \to I$ is continuous. If I is closed and bounded, then so is $f(I)$. In particular, f is bounded on I and has a largest and a smallest value.*

(vii) *If $f \in \mathcal{C}\langle a, b\rangle$, then there exist polynomial functions P_n on $\langle a, b\rangle$ such that $\lim_{n\to\infty} P_n = f$ uniformly. (Approximation theorem of Weierstrass.)*

(viii) *If $f \in \mathcal{C}$, then f has an antiderivative F, f is Riemann integrable over $\langle a, b\rangle$ for every $a, b \in I$ with $a < b$ and*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

(Fundamental theorem of calculus.)

A set $S \subseteq \mathbb{R}^2$ is called *arcwise connected* if for every $v, w \in S$ there is a continuous $\phi\colon [0, 1] \to S$ such that $\phi(0) = v$ and $\phi(1) = w$. [3]

Theorem. *(8.2) Let $f\colon [a, b] \to \mathbb{R}$ and let $\Gamma_f := \{(x, f(x)) : x \in [a, b]\}$ be the graph of f. Then the following conditions are equivalent.*
(α) *f is continuous.*
(β) *Γ_f is compact.*
(γ) *Γ_f is arcwise connected.*

3.9 Darboux continuous functions

Definition. (9.1) Let $f\colon [a, b] \to \mathbb{R}$. f is called *Darboux continuous* if for any p, q with $a \le p < q \le b$ and any $c \in \mathbb{R}$ between $f(p)$ and $f(q)$ there is an s between p and q such that $f(s) = c$.

Continuous functions and derivatives are Darboux continuous.
[4]

Theorem. *(9.4) Let $f\colon [a, b] \to \mathbb{R}$. If the graph Γ_f of f is connected, then f is Darboux continuous.*

Theorem. *(9.5) Every $f\colon \mathbb{R} \to \mathbb{R}$ is the difference of two Darboux continuous functions.*

Notes to Section 9 For a bibliography, see [BC] and [BCW].

3.10 Semicontinuous functions

Definition. (10.1) Let $f\colon I \to \mathbb{R}$.

f is called *lower semicontinuous* if for every $p \in I$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in I$, $|x - p| < \delta$ implies $-\varepsilon < f(x) - f(p)$.

[3] TODO arcwise connected vs. path connected
[4] TODO Theorem 9.2, Corollary 9.3

f is called *upper semicontinuous* if for every $p \in I$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in I$, $|x-p| < \delta$ implies $f(x) - f(p) < \varepsilon$.

$\mathcal{C}^+(I) =$ all lower semicontinuous functions on I

$\mathcal{C}^-(I) =$ all upper semicontinuous functions on I

Theorem. *(10.2) Let $f, g\colon I \to \mathbb{R}$, $A \subseteq I$. Then*
(i) $f \in \mathcal{C}^+$ if and only if for every $s \in \mathbb{R}$ the set $\{x; f(x) > s\}$ is open in I (i.e. is the intersection of I and an open set).
$f \in \mathcal{C}^-$ if and only if for every $s \in \mathbb{R}$ the set $\{x; f(x) < s\}$ is open in I.
(ii) $\xi_A \in \mathcal{C}^+$ if and only if A is open in I.
$\xi_A \in \mathcal{C}^-$ if and only if A is closed in I.
(iii) If $f, g \in \mathcal{C}^+$ and $\lambda \geq 0$ then $f + g \in \mathcal{C}^+$ and $\lambda f \in \mathcal{C}^+$.
If $f, g \in \mathcal{C}^-$ and $\lambda \geq 0$ then $f + g \in \mathcal{C}^+$ and $\lambda f \in \mathcal{C}^-$.

Theorem. *(10.3) Let $\mathcal{S} \subset \mathcal{C}^+$.*
(i) If $h(x) := \sup\{f(x); f \in S\}$ exists for all x, then $h \in \mathcal{C}^+$.
(ii) If $f, g \in \mathcal{C}^+$ then $f \wedge g \in \mathcal{C}^+$.
(In other words, $\mathcal{C}^+$ is closed under arbitrary suprema and finite infima.)

Definition. (10.4) Let $f\colon I \to \mathbb{R}$ be a bounded function. We define

$$f^{\uparrow} = \sup\{g(x); g \leq f; g \in \mathcal{C}^+\}$$
$$f^{\downarrow} = \inf\{h(x); h \geq f; h \in \mathcal{C}^-\}$$

Theorem. *(10.5) Let f and g be bounded functions on an interval I. Then*
- *(i)* $f^{\uparrow} \leq f \leq f^{\downarrow}$, $f^{\uparrow} \in \mathcal{C}^+$, $f^{\downarrow} \in \mathcal{C}^-$
- *(ii) If $h \leq f$, $h \in \mathcal{C}^+$, then $h \leq f^{\uparrow}$. If $j \geq f$, $j \in \mathcal{C}^-$, then $j \geq f^{\downarrow}$.*
- *(iii) If $f \leq g$, then $f^{\uparrow} \leq g^{\uparrow}$ and $f^{\downarrow} \leq g^{\downarrow}$.*
- *(iv)* $\inf_{x \in I} f^{\downarrow}(x) \geq \inf_{x \in I} f(x) = \inf_{x \in I} f^{\uparrow}(x)$.
  $\sup_{x \in I} f^{\downarrow}(x) = \sup_{x \in I} f(x) \geq \sup_{x \in I} f^{\uparrow}(x)$.
- *(v) f is lower semicontinuous if and only if $f = f^{\uparrow}$.*
  f is upper semicontinuous if and only if $f = f^{\downarrow}$.
  f is continuous if and only if $f^{\uparrow} = f^{\downarrow}$.
- *(vi) For every $t \in I$,*

$$f^{\uparrow}(t) = \lim_{n\to\infty} \inf\{f(x) : x \in I, |x-t| \leq 1/n\}$$
$$f^{\downarrow}(t) = \lim_{n\to\infty} \sup\{f(x) : x \in I, |x-t| \leq 1/n\}$$

Theorem (Baire)**.** *(10.6) Let I be an interval. Let $f\colon I \to \mathbb{R}$ be bounded below. Then the following two conditions are equivalent.*

a) *f is lower semicontinuous.*

b) *There is an increasing sequence of continuous functions on I, tending to f pointwise.*

3.11 Functions of the first class of Baire

$\mathscr{B}^1(X)$ = *functions of the first class of Baire* = pointwise limits of sequences of continuous functions.

Theorem. *(11.2) Let I be an interval. Let $f, g\colon I \to \mathbb{R}$ be of the first class. Then*

- *$f, g \in \mathscr{B}^1$ and $\lambda f \in \mathscr{B}^1$ for every $\lambda \in \mathbb{R}$. Furthermore $fg \in \mathscr{B}^1$. ($\mathscr{B}^1$ is an algebra of functions.)*
- *$f \vee g \in \mathscr{B}^1$ and $f \wedge g \in \mathscr{B}^1$. ($\mathscr{B}^1$ is a lattice.)*
- *If $h\colon \mathbb{R} \to \mathbb{R}$ is continuous, then $h \circ f \in \mathscr{B}^1$.*
- *If J is an interval and $\sigma\colon J \to I$ is continuous, then $f \circ \sigma \in \mathscr{B}^1(J)$.*

Theorem. *(11.3) Let I be an interval and let $f \in \mathscr{B}^1(I)$. Then for every open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is an F_σ (See Theorem 11.12 for the converse.)*

Theorem. *(11.4) Let I be an interval and let $f \in \mathscr{B}^1(I)$. Then the points of discontinuity of f form a meagre F_σ. In particular the points of continuity form a dense set.* [5] [6] [7]

Corollary. *(11.5) $\xi_{\mathbb{Q}}$ is not a function of the first class.*

Theorem. *(11.6) Let $X \subseteq \mathbb{R}$. Then ξ_X, as a function on $\mathbb{R}$, is of the first class if and only if X is both an F_σ and a G_δ.*

Theorem. *(11.7) Let $X \subseteq \mathbb{R}$. Then $\mathscr{B}^1(X)$ is uniformly closed, i.e. if $f_1, f_2, \dots \in \mathscr{B}^1(X)$ and $f := \lim\limits_{n\to\infty} f_n$ uniformly, then $f \in \mathscr{B}^1(X)$.*

Theorem. *(11.8) Let I be an interval. Every function $I \to \mathbb{R}$ that has only countably many points of discontinuity is of the first class.*

(The converse is false, as the example $\xi_{\mathbb{D}}$ shows.)

Corollary. *(11.9) Every left or right continuous function is of the first class of Baire.*

Theorem. *(11.10) Let I be an interval. Let $f\colon I \to \mathbb{R}$ be such that its graph is a closed subset of $\mathbb{R}^2$. Then f is a difference of two semicontinuous functions and therefore is of the first class.*

Lemma. *(11.11) Let $A_1, A_2, \dots, A_N$ be F_σ-sets whose union is $\mathbb{R}$. Then there exist pairwise disjoint F_σ-sets $P_1, P_2, \dots, P_N$ such that $P_1 \cup \dots \cup P_N = \mathbb{R}$ and $P_i \subset A_i$ for each i.*

Theorem (Lebesgue)**.** *(11.12) Let I be an interval and let $f\colon I \to \mathbb{R}$. Then f is of the first class if and only if for every open subset U of $\mathbb{R}$, $f^{-1}(U)$ is an F_σ.*

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[5] The same is true for a map $f\colon X \to Y$, where X, Y are metrizable, Y is separable. For the second part to hold we need additionally that X is completely metrizable. See [K, Theorem 24.14].

[6] The proof in [vRS] uses $\overline{f^{-1}(V)} \setminus \operatorname{Int} f^{-1}(V)$ and proof in uses $f^{-1}(V) \setminus \operatorname{Int} f^{-1}(V)$; where V runs over elements of a countable base of Y. Which of them is correct?

[7] Wikipedia article on Baire spaces claims that this is true if X is a Baire space.

[8] [9] [10]

Corollary. *(11.13) Let $f\colon I \to \mathbb{R}$. Suppose there exist functions $g_1, g_2, \dots,$ $h_1, h_2, \dots$ of the first class such that $g_1 \geq g_2 \geq \dots$, $\lim\limits_{n\to\infty} g_n = f$, $h_1 \leq h_2 \leq \dots$, $\lim\limits_{n\to\infty} h_n = f$. Then $f \in \mathscr{B}^1$.*

Baire has given the following characterization of the elements $\mathscr{B}^1$: A function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to $\mathscr{B}^1$ if and only if for every nonempty closed subset A of $\mathbb{R}$ the restriction of f to A is continuous at some point of A. (We give a proof in Appendix C.)

3.12 Riemann integrable functions

Theorem. *(12.1) Let $f\colon \langle a, b\rangle \to \mathbb{R}$ be bounded. Then the following conditions are equivalent*

a) *f is Riemann integrable.*

b) *The set of points of discontinuity of f is a nullset. (f is continuous almost everywhere.)*

4 Differentiation

4.13 Differentiable functions

$\mathscr{D}$= all differentiable functions on an interval I

Theorem. *(13.1) Let I be an interval.*

(i) If $f, g \in \mathscr{D}$ and $\lambda, \mu \in \mathbb{R}$ then $\lambda f + \mu g \in \mathscr{D}$ and $fg \in \mathscr{D}$. ($\mathscr{D}$ is an algebra of functions.) If $f \in \mathscr{D}$, then f is continuous.

(ii) If $f \in \mathscr{D}$ and $f(x) \neq 0$ for all $x \in I$, then $1/f \in \mathscr{D}$.

(iii) If $f \in \mathscr{D}$, if J is an interval such that $f(I) \subset J$ and if $g\colon J \to \mathbb{R}$ is differentiable, then $g \circ f \in \mathscr{D}$.

(iv) If f is a differentiable bijection of I onto an interval J and if $f'(x) \neq 0$ for all $x \in I$, then the inverse map $f^{-1}\colon J \to I$ is differentiable.

(v) If $p, q \in I$, $p < q$ and $f \in \mathscr{D}$, then there is a $\xi \in (p, q)$ such that $f(q) - f(p) = (q - p)f'(\xi)$ (mean value theorem).

(vi) If $f \in \mathscr{D}$ and $f'(\xi) > 0$ for some $\xi \in I$, then f is increasing at ξ.

(vii) If $f \in \mathscr{D}$ and $f' \geq 0$, then f is increasing on I.
If $f \in \mathscr{D}$ and $f' = 0$, then f is constant.
If $f \in \mathscr{D}$ and $f' > 0$, then f is strictly increasing.

[8] TODO One implication ($F_\sigma \Rightarrow \mathscr{B}^1$) is true for $f\colon X \to Y$, X, Y metric, separable and $Y = \mathbb{R}$ or Y zero-dimensional, [K, Theorem 24.10]

[9] Counterexample before [K, Theorem 24.10] ???

[10] TODO This seems to be true for $f\colon X \to Y$ where X, Y are metric spaces, Y separable. See [K, Thereom 24.3]; although he has this as the definition of $\mathscr{B}^1(X)$.

4.14 Derivatives

Example. (14.1) Let $j \in \mathcal{D}'\langle 0, \infty)$ and $j(x+1) = j(x)$. Let J be the antiderivative of j and $A := J(1) - J(0)$. Then the function

$$h(x) = \begin{cases} j(x^{-1}) & \text{if } 0 < x \leq 1, \\ A & \text{if } x = 0, \end{cases}$$

is differentiable on $\langle 0, 1\rangle$.

Theorem. *(14.2) $\mathcal{D}'$ is uniformly closed, that is, if $\lim_{n\to\infty} f_n = f$ uniformly and if each f_n has an antiderivative, then so has f.*

4.15 The fundamental theorem of calculus

Theorem. *(15.1) Let $f \in \mathcal{D}[a, b]$.*
(i) If $f'(x) > 0$ for all $x \in \langle a, b\rangle$, then f is strictly increasing.
(ii) If $f'(x) \geq 0$ for all $x \in \langle a, b\rangle$, then f is increasing.
(iii) If $f'(x) = 0$ for all $x \in \langle a, b\rangle$, then f is constant.

Corollary. *(15.2) Let $f \in \mathcal{D}[a, b]$. Let $A, B \in \mathbb{R}$ be such that $A \leq f' \leq B$. Then $A(b-a) \leq f(b) - f(a) \leq B(b-a)$.*

Definition. (15.4) Let I be an interval and let $f\colon I \to \mathbb{R}$. For $x \in I$ we define elements $D^+f(x)$ and $D^-f(x)$ of $\mathbb{R} \cup \{\pm\infty\}$ by

$$D^+f(x) = \limsup_{y\to x} \frac{f(y) - f(x)}{y - x}, \qquad D^-f(x) = \liminf_{y\to x} \frac{f(y) - f(x)}{y - x}$$

Theorem. *(15.5) Let $g\colon \langle a, b\rangle \to \mathbb{R}$ be Riemann integrable. Then the function $Jg\colon x \mapsto \int_a^x d(t)\,\mathrm{d}t$ $(x \in \langle a, b\rangle)$ is differentiable almost everywhere on $\langle a, b\rangle$ and $(Jg)' = g$ a.e. on $\langle a, b\rangle$.*

5 Borel measurability

5.16 The classes of Baire

$\mathcal{C} \subset \mathcal{B}^1 \subset \mathcal{B}^2 \subset \mathcal{B}^3 \subset \dots$

We know that the inclusions $\mathcal{C} \subset \mathcal{B}^1$ and $\mathcal{B}^1 \subset \mathcal{B}^2$ are strict.[11]

For ordinals: $\mathcal{B}^{\alpha+1} = (\mathcal{B})^*$ and for limit ordinals $\mathcal{B}^\alpha = \bigcup_{\beta<\alpha} \mathcal{B}^\beta$.

Definition. (16.2) A set $\mathcal{F}$ of functions $\mathbb{R} \to \mathbb{R}$ is called an L-set if
(i) $\mathcal{C} \subseteq \mathcal{F}$;
(ii) if $f_1, f_2, \dots \in \mathcal{F}$ and $f = \lim_{n\to\infty} f_n$, then $f \in \mathcal{F}$

[11] For example $\xi_{\{0\}}$ is in $\mathcal{B}^1$, but it is not continuous. The function $\xi_{\mathbb{Q}}$ is not a function of the first class (Corollary 11.5), but it can be obtained as a limit of ξ_{F_n}, where F_n is a finite set.

The collection of all functions $\mathbb{R} \to \mathbb{R}$ is an L-set. By $\mathcal{B}$ we denote the intersection of all L-sets. This $\mathcal{B}$ is itself an L-set and it is contained in every L-set. The elements of $\mathcal{B}$ are called *Borel functions* or *Borel measurable functions.*

$f, g \in \mathcal{B} \Rightarrow f+g, fg, f \vee g, f \wedge g \in \mathcal{B}$

Theorem. *(16.4) $\mathcal{B}$ is a vector space and a ring containing $\mathcal{C}(\mathbb{R}), \mathcal{B}^1, \mathcal{B}^2, \dots$ If $f \in \mathcal{B}$ and $g \in \mathcal{B}$, then $f \vee g \in \mathcal{B}$, $f \wedge g \in \mathcal{B}$, $f \circ g \in \mathcal{B}$, $|f| \in \mathcal{B}$. If $f_1, f_2, \dots \in \mathcal{B}$ and if $\lim_{n\to\infty} f_n(x)$ exists for all x, then $\lim_{n\to\infty} f_n \in \mathcal{B}$. If $f_1, f_2, \dots \in \mathcal{B}$ and if $\sup_{n\in\mathbb{N}} f_n(x)$ is finite for all x, then $\sup_{n\in\mathbb{N}} f_n \in \mathcal{B}$. (In fact, $\sup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} f_1 \vee f_2 \vee \dots \vee f_n$.)*

Definition. (16.5) A subset E of $\mathbb{R}$ is a *Borel set* (is *Borel measurable*) if $\xi_E \in \mathcal{B}$. We denote the collection of all Borel sets by Ω.

Theorem. *(16.6) (i) $\emptyset, \mathbb{R} \in \Omega$.*
(ii) If $A \in \Omega$ and $B \in \Omega$, then $A \cup B \in \Omega$, $\mathbb{R} \setminus A \in \Omega$, $\mathbb{R} \setminus B \in \Omega$.
(iii) If $A_1, A_2, \dots \in \Omega$, then $\bigcup_n A_n \in \Omega$ and $\bigcap_n A_n \in \Omega$.
(iv) All open sets and all closed sets are elements of Ω (see Theorem 10.2(i)).

Theorem. *(16.7) Let $f\colon \mathbb{R} \to \mathbb{R}$. The conditions (α)–(ϵ) are equivalent.*
(α) *$f \in \mathcal{B}$.*
(β) *For every $a \in \mathbb{R}$, $\{x; f(x) \geq a\} \in \Omega$.*
(γ) *For every $a \in \mathbb{R}$, $\{x; f(x) > a\} \in \Omega$.*
(δ) *For every open set U, $f^{-1}(U) \in \Omega$.*
(ϵ) *For every $E \in \Omega$, $f^{-1}(E) \in \Omega$.*

For a set $\mathcal{A}$ of functions on U we denote by $\mathcal{A}^*$ the set of all functions that can be written as $\lim_{n\to\infty} f_n$ for certain $f_1, f_2, \dots \in \mathcal{A}$. Thus, $\mathcal{C}^* = \mathcal{B}^1$, $\mathcal{B}^{1*} = \mathcal{B}^2, \dots$

Let $\mathcal{A}$ be a set of functions on $\mathbb{R}$. A function $F\colon \mathbb{R}^2 \to \mathbb{R}$ is said to be a catalogue of $\mathcal{A}$ if
(i) F is Borel measurable,
(ii) for every $f \in \mathcal{A}$ there is an $s \in \langle 0, 1\rangle$ such that $f(x) = F(x, s)$ for all $x \in \mathbb{R}$.

Lemma. *(16.9)*

(i) If $\mathcal{A}_1, \mathcal{A}_2, \dots$ have catalogues, then so does $\bigcup_n \mathcal{A}_n$.

(ii) If $\mathcal{A}$ has a catalogue, then so does $\mathcal{A}^*$.

(iii) $\mathcal{C}$ has a catalogue.

(iv) $\mathcal{B}$ does not have a catalogue.

(v) If $\mathcal{A} \supseteq \mathcal{C}$ and $\mathcal{A}$ has a catalogue, then $\mathcal{A}^* \neq \mathcal{A}$.

5.17 Transfinite construction of the Borel functions

Corollary. *(17.12.) $\mathscr{B}$ has the cardinality of continuum.*

5.18 Analytic sets

[12]

In this section, equivalence of the following conditions for subsets of $\mathbb{R}$ will be shown:
(α): A is the image of a Borel set E under a Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$.
(β): A is the image of a G_δ-set E under a Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$.
(γ): A is the image of a G_δ-set E under a Borel measurable function $f\colon E \to \mathbb{R}$.
(δ): There exists a continuous map of $\mathbb{R}\setminus\mathbb{Q}$ onto A.
(ε) There exists a continuous map of $\mathscr{N}$ onto A.
$\mathscr{N} = \mathbb{N}^\mathbb{N}$ with the product topology (=pointwise convergence). [13]

Definition. (18.3) We call a subset A of $\mathbb{R}$ analytic if either $A = \emptyset$ or there exists a continuous surjection $\mathscr{N} \to A$.

Examples: $(0,1\rangle$, $\langle a,b\rangle$, $\mathbb{R}$, $\mathbb{N}$ and any countable subset of $\mathbb{R}$

Theorem. *(18.6) Let $A_1, A_2, \dots$ be analytic subsets of $\mathbb{R}$. Then their union and intersection are analytic.*

Lemma. *(18.7) Let S be a nonempty closed subset of $\mathscr{N}$. Then there exists a continuous surjection $F\colon \mathscr{N} \to S$ with $f(a) = a$ for all $a \in \mathscr{N}$.*

Theorem. *(18.8) All Borel subset of $\mathbb{R}$ are analytic (and have analytic complements).*

Lemma (Separation lemma)**.** *(18.9) If A and B are disjoint analytic sets, then there exists a Borel set E with $A \subset E$, $B \subset \mathbb{R}\setminus E$.*

Theorem. *(18.10) A subset A of $\mathbb{R}$ is Borel if and only if both A and its complement are analytic.*

Theorem (Sierpiński)**.** *(18.12) A function $\mathbb{R} \to \mathbb{R}$ is Borel measurable if and only if its graph is a Borel subset of $\mathbb{R}^2$.*

Corollary. *(18.13) If $f\colon \mathbb{R} \to \mathbb{R}$ is bijective and Borel measurable, then the inverse map f^{-1} is also Borel measurable.*

Theorem. *(18.14) Let A be an analytic subset of $\mathbb{R}$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be Borel measurable. Then $f[A]$ is analytic.*

Theorem. *(18.15) (i) Let π_2 be the second coordinate map $\mathbb{R}^2 \to \mathbb{R}$. A subset X of $\mathbb{R}$ is analytic if and only if there is a G_δ-subset A of $\mathbb{R}^2$ with $X = \pi_2[A]$.*
(ii) There exists a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that the analytic subsets of $\mathbb{R}$ are just the sets $g[B]$ where B runs through the G_δ-subsets of $\mathbb{R}$.

Theorem (Sierpiński)**.** *(18.16) Not all analytic sets are Borel.*

[12] TODO I should find corresponding results for Polish spaces; maybe in [K], and mention them in footnotes.

[13] This space is usually called *Baire space*. It is first countable, completely metrizable; i.e., it is a Polish space.

Lemma. *(18.18) There exists a subset* W *of* $\mathscr{N}^3$ *such that*
(i) for every continuous $F\colon \mathbb{N} \to \mathbb{N}$ *there exists an* $s \in \mathscr{N}$ *such that the graph of* F *is just* $\{(x,y) \in \mathscr{N}^2; (x,y,s) \in W\}$.
(ii) W *is closed*

Corollary. *(18.21) The image* $f[X]$ *of a Borel subset* X *of* $\mathbb{R}$ *under a Borel measurable function* f *need not be a Borel set.*

6 Integration

6.19 The Lebesgue integral

[14]

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *Lebesgue integrable* if there exist $\phi_1, \phi_2 \dots \in \mathscr{C}_c$ such that $\sum_{i=1}^{\infty} \int |\phi_i|$ is finite while $\sum_{i=1}^{n} \phi_i = f$ almost everywhere. Clearly, the Lebesgue integrable functions form a vector space which contains $\mathscr{C}_c$. We denote this vector space by $\mathscr{L}$.

Theorem. *(19.9) If* $f \in \mathscr{L}$*, then* $|f| \in \mathscr{L}$ *and* $|\int f| \le \int |f|$*. If* $f, g \in \mathscr{L}$*, then* $f \wedge g \in \mathscr{L}$ *and* $f \vee g \in \mathscr{L}$.

Lemma. *(19.10) Let* $f \in \mathscr{L}$ *and* $\varepsilon > 0$*. Then there exists a* $\Phi \in \mathscr{C}_c$ *such that* $\int |f - \Phi| < \varepsilon$*. In fact, there exist* $\Phi, \phi_1, \phi_2, \dots \in \mathscr{C}_c$ *with* $f = \Phi + \sum\limits_{i=1}^{\infty} \phi_i$ *a.e. and* $\int |f - \Phi| \le \sum\limits_{i=1}^{\infty} \int |\phi_i| \le \varepsilon$.

Theorem. *(19.11) Let* $f_1, f_2, \dots \in \mathscr{L}$ *be such that* $\sum\limits_{n=1}^{\infty} |f_n| < \infty$*. Then the series* $\sum\limits_{n=1}^{\infty} f_n$ *converges a.e. If* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $f = \sum\limits_{n=1}^{\infty} f_n$ *a.e., then* $f \in \mathscr{L}$ *and* $\int f = \sum_{n=1}^{\infty} \int f_n$.

Corollary. *(19.12) (i) If* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $f = 0$ *a.e., then* $f \in \mathscr{L}$ *and* $\int f = 0$*. Conversely, if* $f \in \mathscr{L}$*,* $f \ge 0$ *and* $\int f = 0$*, then* $f = 0$ *a.e.*
(ii) Let $X \subseteq \mathbb{R}$*. Then* X *is a null set if and only if* $\xi_X \in \mathscr{L}$*,* $\int \xi_X = 0$.

Theorem (Monotone convergence, Levi's theorem)**.** *(19.13) Let* $f_1, f_2, \dots \in \mathscr{L}$ *be such that either* $f_1 \le f_2 \le \dots$ *a.e. or* $f_1 \ge f_2 \ge \dots$ *a.e. and such that the sequence* $\int f_1, \int f_2, \dots$ *is bounded. Then the sequence* $f_1, f_2, \dots$ *converges a.e. If* $f\colon \mathbb{R} \to \mathbb{R}$ *and if* $f = \lim\limits_{n\to\infty} f_n$ *a.e., then* $f \in \mathscr{L}$ *and* $\int f = \lim\limits_{n\to\infty} \int f_n$.

Theorem (Fatou)**.** *(19.14) Let* $f_1, f_2, \dots \in \mathscr{L}$*,* $f_n \ge 0$ *a.e. for every* n*. Suppose that the sequence* $\int f_1, \int f_2, \dots$ *is bounded. Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be such that* $f = \liminf\limits_{n\to\infty} f_n$ *a.e. Then* $f \in \mathscr{L}$ *and*

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

[14]TODO 19.1, 19.2

(To see that not always $\int f = \lim_{n\to\infty} \int f_n$, choose $g \in \mathscr{C}_c$, $g \geq 0$ and set $f_n(x) := g(x+n)$.) [15]

Theorem (Dominated convergence, Lebesgue's theorem)**.** *(19.15) Let $g \in \mathscr{L}$, $g \geq 0$ a.e. Let $f_1, f_2, \dots \in \mathscr{L}$ be such that $|f_n| \leq g$ a.e for all n. Let $f \colon \mathbb{R} \to \mathbb{R}$, $f = \lim_{n\to\infty} f_n$ a.e. Then $f \in \mathscr{L}$ and $\int f = \lim_{n\to\infty} \int f_n$.*

Corollary. *(19.16) If $f \colon \mathbb{R} \to \mathbb{R}$ is a Borel function, if $g \in \mathscr{L}$ and if $|f| \leq g$, then $f \in \mathscr{L}$*

6.20 Lebesgue measurability

Theorem. *(20.9) For $E \subseteq \mathbb{R}$ the following statements are equivalent.* [16]

6.21 Absolute continuity

Definition (Lusin)**.** (21.8) A function $f \colon \langle a,b\rangle \to \mathbb{R}$ is said to have the *property (N)* if for every nullset $N \subseteq \langle a,b\rangle$ its image set $f[X]$ is a null set.

Theorem. *(21.9) Every differentiable function on $\langle a,b\rangle$ has the property (N).*

Theorem. *(21.10) Let $f \colon \langle a,b\rangle \to \mathbb{R}$ be continuous. Suppose that $f'(x) > 0$ (or $f'(x) \geq 0$, $f'(x) = 0$) for almost every $x \in \langle a,b\rangle$ and that f has the property (N). Then f is strictly increasing (or increasing, constant).*

Corollary. *(21.11) Let $f \colon \langle a,b\rangle \to \mathbb{R}$ be differentiable and let $f'(x) > 0$ (or $f'(x) \geq 0$, $f'(x) = 0$) for almost every $x \in \langle a,b\rangle$. Then f is strictly increasing (or increasing, constant).*

Theorem (Vitali-Banach)**.** *(21.21) Let $f \colon \langle a,b\rangle \to \mathbb{R}$.*

- *(i) The following conditions are equivalent:*
  - (α) *f is absolutely continuous.*
  - (β) *f is continuous and of bounded variation and f has the property (N).*
  - (γ) *f is an indefinite integral of a Lebesgue integrable function.*
- *(ii) If f satisfies these conditions, then it is differentiable a.e. on $\langle a,b\rangle$, f' is Lebesgue integrable and f is an indefinite integral of f'.*

[15] My note: I had problems remembering the direction of the inequality in the Fatou's lemma. A mnemonic which helps me to remember it is that integral behaves similarly as sum and we have

$$\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$$

and

$$\sum_{k=1}^{n} \liminf x^{(k)} \leq \liminf \sum_{k=1}^{n} x^{(k)},$$

where each $x^{(k)}$ is a sequence. (The whole inequality is $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n) \leq \liminf x_n + \limsup y_n \leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$.)
Some other tricks to remember the direction of the inequality can be found here: `http://math.stackexchange.com/questions/242920/tricks-to-remember-fatous-lemma`

[16] TODO regularity of Lebesgue measure

6.22 The Perron integral

If $u\colon \langle a,b\rangle \to \mathbb{R}$ and $x,y \in \langle a,b\rangle$ we define

$$\big|_a^b u = u(y) - u(x).$$

Let $f\colon \langle a,b\rangle \to \mathbb{R}$. A continuous function u on $\langle a,b\rangle$ is called an *upper function of f* if $D^- u \geq f$. A continuous function v on $\langle a,b\rangle$ is called a *lower function of f* if $D^+ v \leq f$.

Definition. (22.1) f is said *Perron integrable over* $\langle a,b\rangle$ if for every $\varepsilon > 0$ there exists an upper function u and a lower function v for which $|_u^a b \leq |_v^a b + \varepsilon$. For such Perron integrable f,

$$\sup\left\{\big|_a^b v\,;v \text{ is a lower function of } f\right\} = \inf\left\{\big|_a^b u\,;u \text{ is an upper function of } f\right\}$$

This number is then called the *Perron integral of f over* $\langle a,b\rangle$ we denote it by $\mathcal{P}\int_a^b f$. [17]

Corollary. *(22.8) If $f\colon \langle a,b\rangle \to \mathbb{R}$ is Perron integrable and ≥ 0, then f is also Lebesgue integrable.*

6.23 The Stieltjes integral

Appendixes

A. The real number system

B. Cardinalities

C. An uncountable well-ordered set: a characterization of the functions of the first class of Baire

Theorem (Baire)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$. Then f belongs to first Baire class if and only if, for every non-empty closed subset C of $\mathbb{R}$, the restriction of f to $\mathbb{R}$ has a continuity point.*

D. An elementary proof of Lebesgue's density theorem

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[17] TODO

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