van Rooij, Schikhof: A Second Course on Real Functions

Some notes made when reading [vRS]. (E.g. problems in some exercises, some steps in the proofs that were not obvious (for me) immediately.)

Introduction

(6) A discontinuous Riemann integrable function with an antiderivative. $f(x) = \begin{cases} \sin x^{-1} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$

For the function:
$\phi(x) = \begin{cases} x^2 \cos x^{-1} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$
we get $\phi'(x) = \begin{cases} 2x \cos x^{-1} + \sin x^{-1} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$
Now $\phi' = \psi + f$ where $\psi(x) = \begin{cases} 2x \cos x^{-1} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$
ψ is continuous $\Rightarrow$ ψ has an antiderivative

(7) A Riemann integrable, Darboux continuous function that has no antiderivative.

(8) A function that has an antiderivative but is not Riemann integrable. The function h defined by

$$h(x) := \begin{cases} 2x \sin x^{-2} - 2x^{-1} \cos x^{-2} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases} \tag{0.1}$$

is unbounded, hence certainly not Riemann integrable. But it is the derivative of H, where

$$H(x) := \begin{cases} x^2 \sin x^{-2} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases} \tag{0.2}$$

Example of function that is Darboux continuous, but it is neither Riemann integrable nor it has antiderivative

Further examples of strongly Darboux functions

The above example gives a construction of a real function satisfying $f[I] = \mathbb{R}$ for each non-degenerate interval I. I will call such functions *strongly Darboux,* in accordance with [Ci, Section 7.2].[1]

Construction using transfinite induction If we enumerate all sets of the form $(a, b) \times \{c\}$ as $\{(a_\alpha, b_\alpha) \times \{c_\alpha\}; \alpha < c\}$ then we can construct by transfinite induction a set such that all vertical cuts are singletons and all horizontal cuts are dense in the corresponding horizontal line. (We simply choose (x_α, y_α) such that $y_\alpha = c_\alpha$, $x_\alpha \in (a_\alpha, b_\alpha) \setminus \{x_\beta; \beta < \alpha\}$. We add points $(x, 0)$ for all unused x's.) Such construction is given e.g. in [Ci, Theorem 7.2.1].[2]

Additive function, which is strongly Darboux It is possible to construct a function which is strongly Darboux and, moreover, if has the property that $f(x + y) = f(x) + f(y)$. (I.e. it is a solution of Cauchy functional equation.)

Example. The solutions of Cauchy functional equation are precisely the functions $f\colon \mathbb{R} \to \mathbb{R}$ which are linear, if we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. I.e. every such function is uniquely determined if we choose values attained at elements of some Hamel basis B of $\mathbb{R}$ over $\mathbb{Q}$.

If B is any Hamel basis and we choose arbitrary $f(b) \in \mathbb{Q}$, then the resulting function will only have rational values. Thus there are discontinuous additive functions which are not Darboux. The same example is given in [J, Theorem 3].

Now let B be a Hamel basis such that $1 \in B$. It is easy to show that $|B| = \mathfrak{c} = |R|$. Hence there is a bijection $\phi\colon B \to \mathbb{R}$ such that $\phi(1) = 0$. If we choose $f(b) = \phi(b)$, we get an additive function which attains all real values and $f(x + q) = f(x)$ for any $x \in \mathbb{R}$ and $q \in \mathbb{Q}$, i.e. f is constant on every set $x + \mathbb{Q}$. Such function is clearly strongly Darboux. The same example is given in [BC, Example 3.2], [Bo, Theorem 7].

Example. This example is similar to the above example.

We have $|\mathbb{R}/\mathbb{Q}| = |\mathbb{R}|$. We choose any bijection $\psi\colon \mathbb{R}/\mathbb{Q} \to \mathbb{R}$. We also have the canonical homomorphism $\varphi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Q}$. The composition $f = \psi \circ \varphi$ is a which is constant on every set of the form $x + \mathbb{Q}$ and attains all real values. Hence this function is strongly Darboux. [3]

However, the function from this example is not necessarily additive.

[1] Several such examples can be found `http://mathoverflow.net/questions/32126/function-with-range-equal-to-whole-reals-on-every-open-set/` and `http://math.stackexchange.com/questions/21812/`

[2] `http://books.google.com/books?id=tTEaMFvzhDAC&pg=PA106`

[3] This example is from `http://mathoverflow.net/questions/32126/function-with-range-equal-to-whole-reals-on-every-open-set/32128#32128`

1 Monotone functions

1.1 Continuity of monotone functions

Exercise 1.E. In general, the monotone functions on a subset X of $\mathbb{R}$ do not form a vector space. (The sum of two monotone functions need not be monotone.) Let M denote the linear space of functions generated by the monotone functions. Prove that $f \in M$ if and only if f is the difference of two increasing functions.

Exercise 1.O. The set of the monotone functions on $[0, 1]$ contains all polynomial functions of degree ≤ 1. These form a two-dimensional vector space. Does the set of all monotone functions contain a three-dimensional vector space? [4]

Lemma. *Let $f, g\colon [0, 1] \to \mathbb{R}$ be functions such that $f(0) = g(0) = 0$ and the function $af(x) + bg(x)$ is monotone for any $a, b \in R$. Then $f = 0$ or $g = cf$ for some constant $c \in \mathbb{R}$.*

Proof. Let $f \neq 0$. Let us denote the space consisting of all linear combinations of f and g by V. We assume that all functions in V are monotone.

W.l.o.g we can assume that f is non-decreasing. (Otherwise we can use the same proof for $-f$.)

Let us take an $x_0 > 0$ such that $f(x_0) > 0$.

Put

$$c := \frac{g(x_0)}{f(x_0)}$$
$$h(x) := g(x) - cf(x)$$

We have $h(0) = h(x_0) = 0$, which implies $h(t) = 0$ for every $t \in [0, x_0]$. (Since h is monotone.)

a) If $h = 0$ then $g = cf$.

b) Suppose that $h \neq 0$. Then there exists a point y_0 such that $h(y_0) \neq 0$. We know that $y_0 \notin [0, x_0]$

This implies $0 < x_0 < y_0$. We have

$$0 = f(0) < f(x_0) \leq f(y_0)$$
$$0 = h(0) = h(x_0) \neq h(y_0)$$

W.l.o.g we may assume $h(y_0) > 0$. (Otherwise we can work with $-h$.)

b.1) Suppose that $f(x_0) = f(y_0)$ and define $h_1 = f - h$. Clearly $h_1 \in V$, but

$$0 < h_1(x_0) = f(x_0) > h_1(y_1) = f(x_0) - h(y_0),$$

so h_1 is not monotone.

[4] See also `http://math.stackexchange.com/questions/155625/does-there-exist-a-3-dimensional-subspace-of-real-functions-consisting-only-of-m`

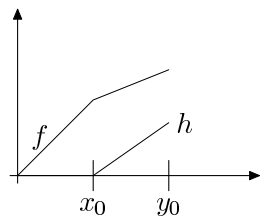

Figure 1: Figure illustrating proof of the lemma on linear combination of monotone functions.

b.2) Now suppose that $f(x_0) < f(y_0)$. (This situation is illustrated in Figure 1.)

In this case we define

$$h_1 := f - 2h\frac{f(y_0) - f(x_0)}{h(y_0)}.$$

Clearly $h_1 \in V$. We have

$$h_1(0) = 0 < h_1(x_0) = f(x_0) > h_1(y_0) = f(y_0) - 2[f(y_0) - f(x_0)] = f(x_0) - [f(y_0) - f(x_0)].$$

So the function h_1 is not monotone. □

Corollary. *If $f, g\colon [0,1] \to \mathbb{R}$ are functions such that the function $af + bg$ is monotone for any $a, b \in \mathbb{R}$, then $f(x) = c$ for some constant $c \in \mathbb{R}$ or $g(x) = cf(x) + d$ for some constants $c, d \in \mathbb{R}$.*

Proof. We apply the above lemma to the functions $f_1(x) = f(x) - f(0)$ and $g_1(x) = g(x) - g(0)$. □

The claim of the exercise follows from this corollary. Indeed, suppose that V is a subspace of $\mathbb{R}^{[0,1]}$ which contains only monotone functions. We can assume that V contains all constant functions, since adding a constant function does not influence monotonicity. The corollary says that if we take two linearly independent functions $1, f \in V$, then all remanding functions in V are linear combinations of 1 and f.

1.2 Indefinite integrals of monotone functions (convex functions)

1.3 Differences of monotone functions

Exercise 3.C. Let $f\colon [0,1] \to \mathbb{R}$ be differentiable with bounded derivative. Then $f \in \mathscr{BV}$. (Corollary. Functions with continuous derivatives are of bounded variation.)

Exercise 3.F. Define $g\colon [0,1] \to \mathbb{R}$ as follows. $g(0) := 0$ and if $x \neq 0$, then $g(x) := x \sin x^{-1}$. Then g is not of bounded variation. The function $x \mapsto g(x^2)$ is differentiable but not in $\mathscr{BV}$. (Compare Exercise 3.C.) The function g^2 is of bounded variation.

1.4 Differentiability of monotone functions

Exercise 4.A. Every countable subset of $\mathbb{R}$ is a null set. (In Example 4.2 we shall present an uncountable null set.) Every subset of a null set is a null set. Unions of countably many null sets are null sets.

Exercise 4.B. The interval $[0,1]$ is not a null set. In fact, if $I_1, I_2, \ldots$ are intervals such that $\sum\limits_{i=1}^{\infty} L(I_i) < 1$, then they do not cover $intrv01$ (Suppose they do; use the Heine-Borel theorem to derive a contradiction.)

Exercise 4.C. Let $E \subset \mathbb{R}$. We say that E has *zero content*[5] if, for every $\varepsilon > 0$, the set E can be covered by finitely many intervals whose total length is less than ε.
(i) Prove that E has zero content if and only if there exists a bounded closed interval $[a,b]$ containing E, such that ξ_E is Riemann integrable over $[a,b]$ and its integral is 0. (In general, for a subset E of a closed interval $[a,b]$ one may define the *content* of E to be the integral of ξ_E when ξ_E is Riemann integrable over $[a,b]$.)
(ii) Prove that a set has zero content if and only if its closure is a bounded null set. (Apply the Heine-Borel covering theorem.)
(iii) Give an example of a bounded null set that does not have zero content.
(iv) Let $f\colon [a,b] \to [0,\infty)$ be Riemann integrable and assume that its integral is 0. Prove that $\{x; f(x) \neq 0\}$ is a null set. (Show that for every $\varepsilon > 0$ the set $\{x; f(x) \geq \varepsilon\}$ has zero content.)

2 Subsets of $\mathbb{R}$

2.5 Small sets

2.6 F_σ-sets and G_δ-sets

Exercise 6.I An F_σ-set is meagre if and only if its interior is empty. [6] Every null F_σ-set is meagre.

[5] Note that the definition is similar to the definition of null set. The difference is the here we only use *finitely many* intervals, in the definition of null set we have *countably many* intervals.

[6] A set with empty interior is sometimes called *nowhere dense* or a *boundary set.* (The name boundary set comes from the fact that such sets are precisely subsets of boundaries of sets.)

Proof. Let A be a F_σ set with empty interior. Then it is union of countably many closed sets with empty interior. (Since $A = \bigcup_n F_n$ and $\operatorname{Int} F_n \subseteq \operatorname{Int} \overline{A} = \emptyset$.) Each F_n is nowhere dense, so the countable union A is meager.

Now let A be a meager F_σ-set. This implies that $A = \bigcup_n A_n$, where $\operatorname{Int} \overline{A_n} = \emptyset$. Simultaneously we have $A = \bigcup_n F_n$ for some closed sets. This implies[7] $A = \bigcup_m \bigcup_n (F_m \cap A_n) = \bigcup_m \bigcup_n (F_m \cap \overline{A_n})$. So A is a countable union of closed nowhere dense sets. (For $B_{mn} = F_m \cap \overline{A_n}$ we have $\overline{B_{mn}} \subseteq \overline{A_n}$ and thus $\operatorname{Int} \overline{B_{mn}} = \emptyset$.) From this we get that $X \setminus B = \bigcap_{m,n} (X \setminus B_{mn})$ is a countable intersection of dense open sets and, by Baire category theorem, it is a dense set. Therefore both B and A have empty interior.

A null set does not contain a non-trivial interval, so it has empty interior. If it is F_σ then, by the first part of the exercise, it is also meager. $\square$

2.7 Behaviour of arbitrary functions

2.7.1 The set of maxima and minima of an arbitrary function

Exercise 7.A. Prove a converse of Theorem 7.2: If $S \subset \mathbb{R}$ is countable, then there exists a (continuous) function $f\colon \mathbb{R} \to \mathbb{R}$ such that S is the set of the local extrema of f.

Additional question: What happens if we want to prescribe sets S_+ and S_- (of local maxima and local minima)? Is there a simple characterization of pairs of sets S_+, S_- for which such function exists? What about continuous functions?

Exercise 7.B: There exists a function $f\colon \mathbb{R} \to \mathbb{R}$, constant on no interval, such that for every $x \in \mathbb{R}$, $f(x)$ is a local extremum. Such an f cannot be continuous.

Example: $f = \chi_{\mathbb{Q}}$.

By Theorem 7.2 $f[\mathbb{R}]$ is countable for any such function. If f is continuous, then $f[\mathbb{R}]$ is an interval. Any non-trivial interval is uncountable, hence the only possibility is that $f[\mathbb{R}]$ is a one-point set and f is constant.

This exercise is solved in [KT, Problem 5.11].

Note: The same argument works for Darboux functions, too. (Theorem 7.2 is valid for arbitrary functions.)

Exercise 7.C: The set $\{a \in \mathbb{R}; f \text{ has a local strict maximum at } a\}$ is countable for any $f\colon \mathbb{R} \to \mathbb{R}$.

This is solved in [KT, Problem 5.10].

[7] The inclusion $\bigcup_n (F_m \cap A_n) \subseteq \bigcup_m \bigcup_n (F_m \cap \overline{A_n})$ is obvious. We also have $\bigcup_m \bigcup_n (F_m \cap \overline{A_n}) \subseteq \bigcup_m F_m = A$.

Exercise 7.G: Let S be an F_σ which does not contain any interval. Then S is a union of closed, nowhere dense sets $C_1, C_2, \dots$ Show that the function $\sum\limits_{n=1}^{\infty} 2^{-n}\xi_{C_n}$ is continuous at every point of $U \setminus S$ and discontinuous at every point of S.

Proof. Note: The existence of sets $C_1, C_2, \dots$ was shown in Exercise 6.I.

Let $x \notin S$. Let $x_n \to x$. This sequence can contain only finitely many elements from each C_k, since C_k is closed and $x \notin C_k$. Hence for any given S we have $x_n \notin \bigcup\limits_{k=1}^{s} C_k$ starting from some n_0. This implies $x_n \le 2^{-s+1}$ for $n \ge n_0$. Hence $f(x)_n$ converges to $0 = f(x)$.

Let $x \in S$. Let I be an arbitrary neighborhood of x. Since S is meager, there exists $y \in I \setminus S$ (by Baire category theorem). For every such y we have $|f(x) - f(y)| \ge 2^{-n}$. Hence f is not continuous at y. □

Proof. It is easy to show that we can get in increasing sequence of set $C_1 \subseteq C_2 \subseteq \dots$ In this case the proof is somewhat simpler since we have $f(x) = \sum_{k \ge n} 2^{-k} = 2^{-(k-1)}$ for $x \in C_n$. Let us discuss this (simpler) case, too.

Let $x \notin S$. Let $x_n \to x$. This sequence can contain only finitely many elements from each C_k, since C_k is closed and $x \notin C_k$. Hence for any given S we have $x_n \notin \bigcup\limits_{k=1}^{s} C_k$ starting from some n_0. This implies $x_n \le 2^{-s+1}$ for $n \ge n_0$. Hence $f(x)_n$ converges to $0 = f(x)$.

If $x \in S$ then x belongs to some C_n. This means that $f(x) = 2^{-n+1}$. Since C_n is nowhere dense, each interval containing x contains also some $y \notin C_n$. Any such y has either $f(y) = 0$ or $f(y) \ge 2^{-n+1} + 2^{-(n+1)}$. Therefore f is not continuous at x. □

Exercise 7.K: [8] Let L be the set of all functions $f\colon [0,1] \to \mathbb{R}$ that have the property that $\lim_{x \to a} f(x)$ exists for all $a \in [0,1]$. Show that:
(i) L is a vector space. Each $f \in L$ is bounded.
(ii) For each $f \in L$, define $f^c(x) := \lim_{y \to x} f(y)$ $(x \in [0,1])$. f^c is continuous.
(iii) '$f^c = 0$' is equivalent to 'there exist $x_1, x_2, \dots$ in $[0,1]$ and $a_1, a_2, \dots$ in U with $\lim_{n \to \infty} a_n = 0$, such that $f(x_n) = a_n$ for every n, and $f = 0$ elsewhere'.
(iv) Describe the general form of an element of L. Show that every $f \in L$ is Riemann integrable.

(i) It is obvious that L is a vector space.

If we fix some $\varepsilon > 0$ then we have for each $a \in [0,1]$ an open neighborhood U_a such that $\operatorname{diam} U_a < \varepsilon$. The sets U_a form an open cover of the compact set $[0,1]$, so we have a finite subcover. Therefore the range of f is covered by finitely many bounded sets and it is bounded itself.

(ii) Let us try to show the continuity of the function f^c defined by $f^c(x) = \lim\limits_{y \to x} f(y)$.

[8] See also `math.stackexchange.com/questions/164946/characterization-of-real-functions-which-have-limit-at-each-p`

Suppose that f^c is not continuous at some point a. Let us denote $b := f^c(a)$. Then there exists a sequence x_n such that $x_n \to a$, $x_n \neq a$ and $f^c(x_n)$ does not converge to b.

By definition of f^c we can find for each x_n a point y_n such that $|y_n - x_n| < 1/2^n$ and $|f(y_n) - g(x_n)| < 1/2^n$. We can assume that, in addition to this, $y_n \neq a$.

For such sequence we have $y_n \to a$ and $f(y_n) \not\to b = f^c(a)$, which contradicts the definition of f^c.

(iii) We want to show $f^c = 0 \Leftrightarrow f$ has the form described in the part (iii).

The implication $\boxed{\Leftarrow}$ is easy: We have $f^c(x) = \lim\limits_{n\to\infty} f(x_n)$ for any sequence $x_n \to x$, $x_n \neq x$. If we choose any ε, there are only finitely many n's such that $|f(x_n)| > \varepsilon$. This implies that the limit must be zero.

$\boxed{\Rightarrow}$ If suffices to show that the set $M_\varepsilon := \{x \in \mathbb{R}; |f(x)| > \varepsilon\}$ is finite for each $\varepsilon > 0$. (Once we have shown this we can get all non-zero points as the union $\bigcap\limits_{n=1}^{\infty} M_{1/n}$, so there is only countably many of them. And they can also be ordered in the required way.)

Suppose that for some $\varepsilon > 0$ the set $M_\varepsilon := \{x \in \mathbb{R}; |f(x)| > \varepsilon\}$ is infinite. Then this set has an accumulation point $a \in [0, 1]$. We have $|f_c(a)| \geq \varepsilon$, contradicting the assumption that f^c is zero.

(iv) Now let f be an arbitrary function from L.

It is relatively easy to see that if we denote $g := f - f^c$, then g^c is identically zero. (I.e., $\lim\limits_{x\to a} g(x) = 0$ for each $a \in [0, 1]$.) Note that $f = f^c + g$.

So we get that each function from L is a sum of a continuous function which has the form described in the part (iii) of this exercise.

Now to show that f is Riemann integrable we could use Lebesgue's criterion for Riemann integrability and the fact that every countable set has Lebesgue measure zero.

But this can be also shown directly from the definition: For any given $\varepsilon > 0$ we have only finitely many points such that $|f(x)| \geq \varepsilon$. If we cover them with small enough intervals, we will get that the Riemann sum over these intervals is in absolute value at most ε. For arbitrary partition containing the small intervals, the absolute value of the Riemann sum over the remaining intervals is at most ε. So we have $|R(f, \Delta)| \leq 2\varepsilon$ whenever the norm of the partition is chosen small enough to ensure that the intervals around the "exceptional" points (the points with $|f(x)| \geq \varepsilon$) are small enough.

2.7.2 The set of points where a continuous function is differentiable

2.7.3 The set of points where a continuous function is increasing

Exercise 7.O: Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable at $c \in \mathbb{R}$. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(c) \right| < \varepsilon$$

for all x, y with $c-\delta < x \le c \le y < c+\delta$ and $x \ne y$.

If $f'(c) = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim\limits_{y\to c} \frac{f(c)-f(y)}{c-y}$, then for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for x and y as above we have

$$\frac{f(x)-f(c)}{x-c} < f'(c)+\varepsilon$$
$$\frac{f(c)-f(y)}{c-y} < f'(c)+\varepsilon$$

If we use the lemma below for the above fractions we get

$$\frac{f(x)-f(c)+f(c)-f(y)}{x-c+c-y} = \frac{f(x)-f(y)}{x-y} < f'(c)+\varepsilon.$$

Similarly we can show

$$\frac{f(x)-f(y)}{x-y} > f'(c)-\varepsilon,$$

which finishes the proof of the above inequality.

Lemma. *Let $a, b, c, d, M, m \in \mathbb{R}$ and $b, d > 0$. Then*

$$\frac{a}{b} < M \wedge \frac{c}{d} < M \Rightarrow \frac{a+c}{b+d} < M$$
$$\frac{a}{b} > m \wedge \frac{c}{d} > m \Rightarrow \frac{a+c}{b+d} > m$$

Proof. The inequalities $ab < M$ and $\frac{c}{d} < M$ are equivalent to

$$a < bM,$$
$$c < dM,$$

and these two inequalities imply $a+c < (b+d)M$ and $\frac{a+c}{b+d} < M$.

The second part is shown in the same way. [9] [10] □

3 Continuity

3.8 Continuous functions

Exercise 8.F (Dini) Let $g, f_1, f_2, \ldots$ be continuous functions on $[a,b]$ such that $f_1(x) \ge f_2(x) \ge \ldots$ and $\lim\limits_{n\to\infty} f_n(x) = g(x)$ for all $x \in [a,b]$. Then $\lim\limits_{n\to\infty} f_n = g$ uniformly. Give an example of a decreasing sequence of continuous functions, tending pointwise (but not uniformly) to a limit function that is not continuous.

[9] We have used similar lemma in [MST, Lemma 2.1]: Let $a_1, a_2, \ldots a_k$ be real numbers and $b_1, b_2, \ldots, b_k$ be positive real numbers. Then

$$\min \frac{a_i}{b_i} \le \frac{a_1+a_2+\cdots+a_k}{b_1+b_2+\cdots+b_k} \le \max \frac{a_i}{b_i}.$$

Moreover, the equalities hold if and only if all quotients a_i/b_i are equal.

[10] See also: `http://math.stackexchange.com/questions/742216/how-to-prove-an-inequality/`

3.9 Darboux continuous functions

Exercise 9.A: Let $f(x) = \sin^{-1} x$ for $x \neq 0$, $f(0) = 1$. Is $f + h$ Darboux continuous for all continuous functions $h\colon \mathbb{R} \to \mathbb{R}$?

The map $(x, y) \mapsto (x, y+h(x))$ is a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. It maps Γ_f to Γ_{f+h}. Since Γ_f is connected, the image Γ_{f+h} under the above homeomorphism is connected, too.

Exercise 9.B: Let $f\colon [a, b] \to \mathbb{R}$. Consider the conditions (δ) and (ε).
(δ) Γ_f is closed and connected.
(ε) f is Darboux continuous and Γ_f is closed.
Show that both (δ) and (ε) are equivalent to (α), (β) and (γ) of Theorem 8.2 (i.e.equivalent to continuity of f).

Every compact subset of $\mathbb{R}^2$ is closed, every path-connected set is connected. Hence $(\beta) \wedge (\gamma) \Rightarrow (\delta)$.

We have $(\delta) \Rightarrow (\varepsilon)$ by Theorem 9.4. (Connected graph $\Gamma_f \Rightarrow f$ is Darboux continuous.)

We will show that every function fulfilling (ε) is (sequentially) continuous.

Let $x_n \to x$. Suppose, by contradiction, that $f(x_n) \nrightarrow f(x)$. This implies that there is an $\varepsilon > 0$ and a subsequence x_{n_k} such that $|f(x_{n_k}) - f(x)| > \varepsilon$. For the sake of simplicity, denote the subsequence again by x_n. W.l.o.g. let $f(x_n) > f(x)+\varepsilon$. (Clearly, one of the sets $\{n; f(x_n) > f(x)+\varepsilon\}$ and $\{n; f(x_n) < f(x)-\varepsilon\}$ is infinite, we choose the one that is infinite.)

Using Darboux property we get existence of y_n between x_n and x such that $f(y_n) = f(x) + \varepsilon$. I.e. we know that $y_n \to x$ and $f(y_n) \to f(x) + \varepsilon$. Hence the point $(x, f(x) + \varepsilon) \in \overline{\Gamma_f} = \Gamma_f$. This contradicts the fact that the graph Γ_f intersects the vertical line through x only in the point $(x, f(x))$. [11]

Exercise 9.C: (A Darboux continuous g with a disconnected graph)
(i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $f(I) = \mathbb{R}$. Define

$$g(x) := \begin{cases} f(x) & \text{if } f(x) \neq x, \\ x + 1 & \text{if } f(x) = x. \end{cases}$$

Prove that g is Darboux continuous but that $x \mapsto g(x)-x$ is not. Since $g(x) \neq x$ for every x we have that $A := \{(x, y) \in \mathbb{R}^2 : y > x\}$ and $B := \{(x, y) \in \mathbb{R}^2 : y < x\}$ are disjoint open sets covering Γ_g. As both have nonempty intersection with Γ_g, the graph of g is not connected.
(ii) Show that the graphs of the functions g and $x \mapsto g(x)-x$ are homeomorphic. Apparently, Darboux continuity of a function is not a topological property of its graph.

Remark. Note that we show an existence of a Darboux function such that $(\forall x \in \mathbb{R}) g(x) \neq x$. (Functions with this property might sometimes be useful for constructing various counterexamples.)

[11] Example of a function, which has closed graph but is not continuous: $f(x) = \frac{1}{x}$, $f(0) = 0$.

$g(x)$ is strongly Darboux. Let $I = (a, b)$ and $I_1 = (a, (2a + b)/3)$, $I_2 = ((2a + b)/3, (a + 2b)/3)$, $I_3 = ((a + 2b)/3, b)$ be a division of I into three parts. We know that $f[I_1] = \mathbb{R}$, which implies $g[I_1] \supseteq \mathbb{R} \setminus I_1$. (If $x \in I_1$ and $f(x) \notin I_1$ then necessarily $f(x) = g(x)$.) By the same argument $f[I_3] \supseteq \mathbb{R} \setminus I_3$. Hence $f[I] \supseteq f[I_1] \cup f[I_3] \supseteq (\mathbb{R} \setminus I_1) \cup (\mathbb{R} \setminus I_3) = \mathbb{R}$.

$h(x)$ is not Darboux. Obviously $h(x) \neq 0$ for no $x \in \mathbb{R}$.

Γ_g and Γ_h are homeomorphism. The map $(x, y) \mapsto (x, y - x)$ is a linear bijective map, hence it is a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. The restriction to Γ_g is homeomorphism of the graphs of the two functions.

Exercise 9.D. Is every function $f\colon \mathbb{R} \to \mathbb{R}$ a product of two Darboux continuous functions?

The function given by

$$f(x) = \begin{cases} -1 & x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

cannot be obtained as a product of two Darboux function. If we have $f = f_1 \cdot f_2$, then

- $f_1(x) > 0$, $f_2(x) > 0$ for $x \neq 0$;
- $f_1(0) < 0$, $f_2(0) < 0$; i.e., the functions f_1, f_2 attain both positive and negative values;
- There is no point x such that $f_1(x) = 0$ or $f_2(x) = 0$. (And thus f_1, f_2 are not Darboux continuous.)

If the function f only attains positive values, we can use Theorem 9.5 to get $f_{1,2}$ such that $f_1 + f_2 = \ln f$, i.e. $f = e^{f_1} \cdot e^{f_2}$. Hence in this case f can be written as a product of two Darboux functions. (The function e^{f_i} is a composition of two Darboux functions, hence it is Darboux by Corollary 9.3.)

Some characterization of functions that can be obtained as a product of two Darboux functions is given in [Ce].

Exercise 9.E. Show that there exist Darboux continuous functions that do not have antiderivatives.

Let f be any function that does not have antiderivative. (In particular, this works for any $f \notin \mathcal{DC}$.) By Theorem 9.5. we have $f = f_1 + f_2$, where f_1 and f_2 are Darboux. At least one of the functions $f_{1,2}$ does not have an antiderivative.

Exercise 9.F. Le $f\colon [a, b] \to \mathbb{R}$ be Darboux continuous. Show that f and $|f|$ have the same points of continuity.

Obviously, $f(x_n) \to f(x)$ implies $|f(x_n)| \to |f(x)|$, hence $C_f \subseteq C_{|f|}$ for any function f.

Now let us assume that f is not continuous at x, which means that there is a sequence (x_n) such that $x_n \to x$ and $f(x_n) \nrightarrow f(x)$.

If $f(x) = 0$ then we have $|f(x_n)| \nrightarrow |f(x)|$.

If $f(x) > 0$ then there are two possibilities. If $f(x_n) > 0$ for all but finitely many n's, then the convergence character of $(f(x_n))$ and $(|f(x_n)|)$ is the same, hence $|f(x_n)| \nrightarrow |f(x)| = f(x)$.

If $f(x_n) < 0$ for infinitely many n's then we can use the Darboux property for the points x_n and x to get a sequence (y_n), such that $y_n \to x$ and $f(y_n) = 0$. Again, we have $|f(y_n)| = f(y_n) \nrightarrow f(x) = |f(x)|$.

The case $f(x) < 0$ is similar.

Exercise 9.G. Let $f\colon [a,b] \to \mathbb{R}$ be such that $f + h \in \mathcal{DC}$ for all $h \in \mathcal{DC}$. Show that f is constant. (Hint. First prove that f is Darboux continuous. Now suppose that $f([a,b])$ contains an interval I. Let $g\colon \mathbb{R} \to \mathbb{R}$ be as in Exercise 9.C. Show that there exist $p, q \in I$ such that $g(p) > p$ and $g(q) < q$. Define $h := -g \circ f$. Prove that $h \in \mathcal{DC}$ but $f + h \notin \mathcal{DC}$.)

Since $f + 0 \in \mathcal{DC}$, the function f is Darboux continuous.

Now assume that $f([a,b])$ contains a non-trivial bounded interval I.

We will use a function $g\colon I \to \mathbb{R}$ such that $g(x) \neq x$ for $x \in I$ and every real value is attained on every subinterval of I (see the construction given in Exercise 9.C). If we choose any $y > \max I$, then this value must be attained at some point $p \in I$. Obviously $g(p) = y > p$. Similarly we can show that there exists a point q such that $g(q) < q$.

The function $g \circ f$ is a composition of two Darboux continuous functions, hence it is Darboux continuous and $h = -g \circ f$ is Darboux continuous too.

Let p_1, q_1 be such that $f(p_1) = p$, $f(q_1) = q$. Then we have
$f(p_1) + h(p_1) = p + g(p) > 0$
$f(q_1) + h(q_1) = q + g(q) < 0$
For any x we have $f(x) + h(x) = f(x) - g(f(x)) \neq 0$.

Thus $f + h$ is not Darboux continuous.

Exercise 9.H. (i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be Darboux continuous and such that $f(x+)$ and $f(x-)$ exist for every $x \in \mathbb{R}$. Prove that f is continuous.
(ii) Find a Darboux continuous $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x+)$ exists for all $x \in \mathbb{R}$ while g is not continuous.[12]

(i) Suppose that $f(x+) \neq f(x-)$. Construct a sequence $x_n \to x$ such that $f(x_n) \nrightarrow f(x)$.

(ii) $\sin \frac{1}{x}$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$.

Exercise 9.I. Every injective Darboux continuous function is continuous.

injective + Darboux $\Rightarrow$ monotone [13]

Exercise 9.J. TODO

[12] Question: Is it possible to find an example discontinuous at *everywhere*?

[13] See also `http://math.stackexchange.com/questions/83340/intermediate-value-property-problem`, `http://math.stackexchange.com/questions/7172/injective-functions-with-intermediate-value-property-are-continuous-better-proo`

Exercise 9.M. (Another function that maps every interval onto $[0,1]$) For $x \in [0,1]$ let $0.x_1x_2x_3\dots$ be the standard dyadic development of $x - \lfloor x \rfloor$:

$$x_n = \lfloor 2^n x \rfloor - 2\lfloor 2^{n-1} x \rfloor$$

where $\lfloor x \rfloor$ is the entire part of x. Define $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Show that ϕ maps every interval *onto* $[0,1]$. (Hint: First show that $\phi(x) = \phi(y)$ if there exist $p, q \in \mathbb{N}$ such that $x_p = y_q$, $x_{p+1} = y_{q+1}$, $x_{p+2} = y_{q+2}$, etc., so that it suffices to show that ϕ maps $[0,1]$ onto $[0,1]$. Now let $t \in [0,1]$, $t \neq 1$. Find an $x \in [0,1]$ such that $x_1 + \cdots + x_n = \lfloor nt \rfloor$ for every n and prove that $\phi(x) = t$. Finally, find an x with $\phi(x) = 1$.)
[14]

Exercise 9.N. For a function $f\colon [a,b] \to \mathbb{R}$ we define the counting function $N_f\colon \mathbb{R} \to \{0,1,2,\dots,\infty\}$: for every $y \in \mathbb{R}$, $N_f(y)$ will be the number of elements of the set $\{x : f(x) = y\}$.

Prove the following. If $f\colon [a,b] \to \mathbb{R}$ is Darboux continuous but not continuous, then there exists an interval I such that $N_f(y) = \infty$ for all $y \in I$.
[15]

Proof. Suppose that f is not continuous at the point x_0 and denote $y_0 = f(x_0)$. There exists an $\varepsilon > 0$ and such that we can find an x with the property $|f(x) - f(x_0)| > \varepsilon$ arbitrarily close to x_0. W.l.o.g suppose that we can find x such that $f(x) > f(x_0) + \varepsilon$ arbitrarily close to x_0. Then by Darboux property we can find infinitely many points x' with $f(x') = y$ for any $y \in (y_0, y_0 + \varepsilon)$. □

Exercise 9.O. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Darboux continuous function such that for every $y \in \mathbb{R}$ the set $\{x \in \mathbb{R}; f(x) = y\}$ is closed. Show that f is continuous.[16]

Proof. Suppose that f is not continuous at the point x and denote $y = f(x)$. There exists $\varepsilon > 0$ and a sequence $x_n \to x$ such that $|f(x_n) - f(x)| > \varepsilon$ for each n. W.l.o.g. we can assume that $f(x_n) > f(x) + \varepsilon$ for each n. By Darboux property there is a y_n between x_n and x such that $f(y_n) = f(x) + \varepsilon = y + \varepsilon$. I.e. we have that $y_n \to x$, $y_n \in f^{-1}(y + \varepsilon)$ bud $x \notin f^{-1}(y + \varepsilon)$. Thus the set $f^{-1}(y + \varepsilon) = \{x \in \mathbb{R}; f(x) = y + \varepsilon\}$ is not closed. □

[14] See also [KN2, Problem 1.3.29] or `http://math.stackexchange.com/a/44285/`

[15] This function is sometimes called *Banach indicatrix*.

[16] The same problem is given here: `http://math.stackexchange.com/questions/83786/a-question-about-darboux-functions`.

Exercise 9.P. Let $0 < \alpha < 1$. Let $f\colon [0,1] \to \mathbb{R}$, $f(0) = f(1)$.
(i) Show that if α is one of the numbers $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ and if f is continuous, then the graph of f has a horizontal chord α, i.e., there exists $s, t \in [0,1]$ with $f(s) = f(t)$ and $|s - t| = 0$.
(ii) The proof you gave probably relies on Darboux continuity. Prove, however, that the given continuity condition on f may not be weakened to Darboux continuity. (Take $\alpha := \frac{1}{2}$ and start with a function on $(0, \frac{1}{2}]$ that maps every subinterval of $(0, \frac{1}{2}]$ onto $\mathbb{R}$.)
(iii) Now let $\alpha \notin \{\frac{1}{2}, \frac{1}{3}, \dots\}$. Define a continuous function on $[0,1]$ with $f(0) = f(1)$ whose graph has *no* horizontal chord of length α. (Choose f such that $f(x + \alpha) = f(x) + 1$ for $x \in [0, 1 - \alpha]$.)

The result described in this exercise is known as *universal chord theorem* or *horizontal chord theorem.*[17]

Solution. (i) TODO

(ii) TODO

(iii) We want to have $f(x + \alpha) = f(x) + 1$. Notice that this also implies $f(x + k\alpha) = f(x) + k$.

If we define the function on the interval $[0, \alpha]$, then the above condition determines the function f uniquely on the rest of the interval $[0,1]$. We want to have $f(0) = 0$ and $f(\alpha) = f(1)$.

Let us denote $n = \lfloor \frac{1}{\alpha} \rfloor$, i.e., n is the largest integer such that $n\alpha < 1$. Then we have $1 - n\alpha \in (0, \alpha)$. We need to choose $f(1 - n\alpha) = -n$ in order to get $f(1) = 0$. (Notice that this cannot be done if $n\alpha = 1$. It is possible only if $0 < 1 - n\alpha < \alpha$, since the values $f(0)$ and $f(1)$ are already prescribed.)

Then arbitrary function defined as above (i.e., with the prescribed values in the points 0, $1 - n\alpha$, α and extended from $[0, \alpha]$ to the whole interval using $f(x + \alpha) = f(x) + 1$) satisfies the required conditions. (Such function for a specific choice of α is illustrated in Figure 2.) □

3.10 Semicontinuous functions

Exercise 10.E Let $f, g \in \mathscr{C}^+$, $f, g \geq 0$. Then $fg \in \mathscr{C}^+$.

Proof. TODO □

Proof. If we use Exercise 10.G, then the proof is easy:

$$\liminf f(x_n)g(x_n) \geq \liminf f(x_n) \liminf g(x_n) \geq f(x).$$

(The inequality for limes inferior - see, for example, [KN1, Problem 2.4.17] and [KN2, Problem 1.4.9].) □

[17] `https://math.stackexchange.com/q/16374` Universal Chord Theorem

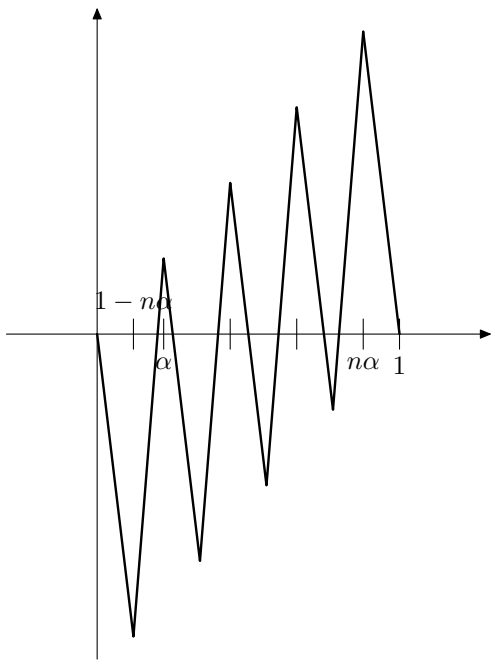

Figure 2: Sketch of a function illustrating Exercise 9.P

Exercise 10.F If $f_1, f_2, \dots \in \mathscr{C}^+$ and $f := \lim_{n\to\infty} f_n$ uniformly, then $f \in \mathscr{C}^+$.

Proof. TODO $\square$

Exercise 10.G Let $f\colon I \to \mathbb{R}$. Then the following are equivalent.
(α) $f \in \mathscr{C}^+$
(β) If $a, x_1, x_2, \dots \in I$ and $a = \lim\limits_{n\to\infty} x_n$ then $f(a) \le \liminf\limits_{n\to\infty} f(x_n)$.

Remark on $\liminf$ **at a point.** There are two "reasonable" definitions of limit inferior at a point:

- $\liminf\limits_{t\to x} f(t) = \lim\limits_{\varepsilon\to 0^+} \inf\{f(t); t \in (x-\varepsilon, x+\varepsilon)\}$;
- $\liminf\limits_{t\to x} f(t) = \lim\limits_{\varepsilon\to 0^+} \inf\{f(t); t \in (x-\varepsilon, x+\varepsilon), t \ne x\}$.

If we use the second definition, then lower semicontinuity is equivalent to $f(x) \le \liminf\limits_{t\to x} f(t)$. If we use the first one, then it is equivalent to $f(x) = \liminf\limits_{t\to x} f(t)$.

Exercise 10.I Let $f \in \mathscr{C}^+[a,b]$. Then f has an (absolute) minimum on $[a,b]$. (Hint. Consider the proof you know for continuous functions.)

Proof. Let $m = \inf_{[a,b]} f(x)$. Suppose that m is not attained at any point. Then there exists a sequence x_n such that $f(x_n) \searrow m$. Since $[a,b]$ is compact, there is a convergent subsequence, we will denote such subsequence by (x_n) again.

So we have $x_n \to x$. Semicontinuity implies $f(x) \le \liminf f(x_n) = m$. Since m is the infimum, we get $f(x) = m$. Contradiction. [18] $\square$

Proof. Let $m = \inf_{[a,b]} f(x)$. By Theorem 10.3 every set of the form $f^{-1}((-\infty, c])$ is closed. If $c > m$ then such set is also non-empty.

We have

$$f^{-1}(m) = f^{-1}(\bigcap_{c>m} (-\infty, c]) = \bigcap_{c>m} f^{-1}((-\infty, c]).$$

This is a system of closed non-empty sets which has finite intersection property. Since we are working in a compact space, the intersection is non-empty. Therefore the infimum is attained. [19] $\square$

Proof. A function $f\colon [a,b] \to \mathbb{R}$ is lower semicontinuous if and only if it is continuous considered as a map to $\mathbb{R}$ with the topology generated by $\{(a,\infty); a \in \mathbb{R}\}$. A continuous image of a compact set is compact. So it suffices to show that every compact set in this topology has smallest element. $\square$

[18] The same proof works in arbitrary compact topological space instead of $[a,b]$, if we work with nets and subnets instead of sequences.

[19] Again, this proof works for arbitrary compact topological space.

Exercise 10.K. Let $f\colon I \to \mathbb{R}$ be a bounded function. For $t \in I$ and $n \in \mathbb{N}$ define $S(n,t) := \sup\{|f(x) - f(y)|; x, y \in I, |x-t| \le 1/n, |y-t| \le 1/n\} = \sup\{f(x); x \in I, |x-t| \le 1/n\} - \inf\{f(x); x \in I, |x-t| \le 1/n\}$. Then $\omega(t) := \lim\limits_{n\to\infty} S(n,t)$ exists for all $t \in I$ It is called the oscillation of f at t. f is continuous at t if and only if $\omega(t) = 0$. Show that $\omega = f^{\uparrow} - f^{\downarrow} \in \mathcal{C}^{-}$. Then $\{x; \omega(x) \neq 0\} = \bigcup_n \{x; \omega(x) \ge 1/n\}$ is an F_σ (We have rediscovered Theorem 7.5.)

Exercise 10.L Let f and g be bounded functions on I. Do we necessarily have $f^{\uparrow\downarrow\uparrow} = f^{\uparrow}$? Do we necessarily have $f^{\uparrow\downarrow\uparrow\downarrow} = f^{\uparrow\downarrow}$? Must $(f+g)^{\uparrow} = f^{\uparrow} + g^{\uparrow}$?

Easy observation: If $f = \chi_A$, then $f^{\downarrow} = \chi_{\overline{A}}$ and $f^{\uparrow} = \chi_{\operatorname{Int} A}$. (It suffices to notice that $\lim_{\varepsilon\to 0^+} \inf_{t\in(x-\varepsilon,x+\varepsilon)} \chi_A(t) = 1 \Leftrightarrow x \in \operatorname{Int} A$ and that $\lim_{\varepsilon\to 0^+} \sup_{t\in(x-\varepsilon,x+\varepsilon)} \chi_A(t) = 0 \Leftrightarrow x \in \overline{A}$.)

Using this observation we have $f^{\uparrow\downarrow\uparrow} \neq f^{\uparrow}$ for any $f = \chi_A$ if $\operatorname{Int}(\overline{\operatorname{Int} A}) \neq A$. (For example, $A = [0,1/2) \cup (1/2,1]$ yields $\overline{\operatorname{Int} A} = [0,1]$ and $\operatorname{Int}(\overline{\operatorname{Int} A}) = (0,1) \neq A$.)

Similarly for $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ (and for the corresponding indicator functions $f = \chi_A$ and $g = \chi_B$) we have $f + g = \chi_A + \chi_B = \chi_{\mathbb{R}}$; thus $(f+g)^{\uparrow} = f + g = \chi_{\mathbb{R}}$ and $f^{\uparrow} = g^{\uparrow} = 0$ (since $\operatorname{Int} A = \operatorname{Int} B = \emptyset$). So for this example we have $(f+g)^{\uparrow} \neq f^{\uparrow} + g^{\uparrow}$.

Note that (from the properties of $\liminf$) we get

$$f^{\uparrow} + g^{\uparrow} \le (f+g)^{\uparrow}.$$

So it only remains to prove

$$f^{\uparrow\downarrow\uparrow\downarrow} = f^{\uparrow\downarrow}.$$

Proof. For any g we have $g^{\uparrow} \le g$ and from this (using monotonicity) we also get $g^{\uparrow\downarrow} \le g^{\downarrow}$. If we apply this inequality to $g = h^{\downarrow}$, we get

$$h^{\downarrow\uparrow\downarrow} \le h^{\downarrow}.$$

(Since $h^{\downarrow\downarrow} = h^{\downarrow}$.)

Applying the last inequality to $h = f^{\uparrow}$ yields

$$f^{\uparrow\downarrow\uparrow\downarrow} \le f^{\uparrow\downarrow}.$$

Similarly, we have $g \le g^{\downarrow}$ and monotonicity give $g^{\uparrow} \le g^{\downarrow\uparrow}$. If we apply this to $g = f^{\uparrow}$, we get

$$f^{\uparrow} \le f^{\uparrow\downarrow\uparrow}.$$

Again using the monotonicity of $^{\downarrow}$-operator, this implies

$$f^{\uparrow\downarrow} \le f^{\uparrow\downarrow\uparrow\downarrow}.$$

$\square$

Exercise 10.M Let $g, f_1, f_2, \dots$ be bounded functions on I such that $\lim_{n\to\infty} f_n = g$ uniformly. Then $\lim_{n\to\infty} f_n^{\uparrow} = g^{\uparrow}$ uniformly.

Proof. $(\forall x \in I)|f_n(x) - g(x)| < \varepsilon \Rightarrow (\forall x \in I)|\liminf_{t\to x} f_n(t) - \liminf_{t\to x} g(t)| \le \varepsilon \Rightarrow (\forall x \in I)|f^{\uparrow}(x) - g^{\uparrow}(x)| \le \varepsilon$ □

Exercise 10.N Let $f\colon I \to \mathbb{R}$. Suppose there exist sequences $h_1 \ge h_2 \ge \dots$ and $g_1 \le g_2 \le \dots$ of continuous functions that both tend to f pointwise. Then f is continuous. (Compare Exercise 8.F.)

Proof. We have $f = \inf h_n$, hence $f \in \mathscr{C}^-$. Similarly, $f = \sup g_n$ implies $f \in \mathscr{C}^+$. □

Dini's theorem (Exercise 8.F) is a special case of this result – if one of the two sequences is constant.

Exercise 10.O. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function whose oscillation (see Exercise 10.K) is everywhere less than 10. Then there is a continuous function $g\colon [a,b] \to \mathbb{R}$ such that $|f(x) - g(x)| < 10$ for all $x \in [a,b]$.

Lemma. *Suppose that the oscillation $\omega_f(x)$ of a function f is smaller than η at each point x of an interval $[c,d]$. Show that there must be a partition $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ of $[c,d]$ so that the oscillation*

$$\omega f([x_{k-1}, x_k]) < \eta$$

on each member of the partition.

Here

$$\omega f(I) = \sup_{x,y\in I} |f(x) - f(y)|$$

See [TBB, Exercise 8.6.6] (this exercise is used in the proof of Riemann-Lebesgue theorem.)

Proof. Since $\omega(x) = \lim_{\varepsilon\to 0^+} \omega f(x-\varepsilon, x+\varepsilon)$, we get that for each x there exists an open interval $I_x = (x-\varepsilon, x+\varepsilon)$ with $\omega(I_x) < \eta$. Now $\{I_x, x \in [c,d]\}$ is an open cover, so it admits a finite subcover. Using that finite subcover we can construct a partition with required property. □

If we have a partition with the above properties, then for each interval of this partition $\sup_{x\in I} f(x) - \inf_{x\in I} f(x) < \eta$. Then we can construct $g(x)$ as a piecewise linear function such that $g(x)$ is between the values $\inf_{x\in I} f(x)$ and $\sup_{x\in I} f(x)$ on each interval of this partition.

Exercise 10.P Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function with only finitely many discontinuities. Then f is the difference of two bounded lower semicontinuous functions.

Proof. If x is a continuity point, then $f^{\uparrow}(x) = f(x) = f^{\downarrow}(x)$. So $g(x) = f^{\uparrow}(x) - f(x)$ is a function which has only finitely many non-zero values and all of them are negative; hence it is lower semicontinuous. Thus we get $f = f^{\uparrow} - g$. $\square$

Exercise 10.Q Let $f\colon [0,1] \to \mathbb{R}$ be monotone. Then $f = g - h$ for certain bounded $g, h \in \mathscr{C}^{+}$. Thus, any function of bounded variation is the difference of two bounded semicontinuous functions.

Exercise 10.R Let $f\colon [0,1] \to \mathbb{R}$ be defined as follows. $f(x) := 0$ if x is irrational, while for rational x, $f(x) := 1/n$, where n is the denominator of the irreducible fraction representing x. Compute $f^{\uparrow}$ and $f^{\downarrow}$.

This function is upper semicontinuous. (See also [KN2, Problem 1.4.14].) Hence $f^{\downarrow} = f$.

$f^{\uparrow}(x) = \liminf_{t \to x} f(x) = 0$.

Exercise 10.S Let f be a function on $[a,b]$ which has a local maximum at every point of $[a,b]$. (Example: the restriction of $\xi_{[1,2]}$ to $[0,2]$.)
(i) Show that f is upper semicontinuous.
(ii) Show that, if f is continuous, then f is constant.
(iii) Show that there exists an interval on which f is constant. (Otherwise, there exist closed intervals $I_1 \supset I_2 \supset \dots$ and points $x_i \in I_i$ such that $f(x_1) > f(x_2) > \dots$ and $f \le f(x_i)$ on I_i for each i.)
(iv) Find a function f on $[0,1]$ that has a local maximum at every point of $[0,1]$ and that has infinitely many values.

Solution. (i) If x is a local maximum then there exist a neighborhood U of x such that $f(y) \le f(x)$ for each $y \in U$. This clearly implies $f(y) < f(x) + \varepsilon$ for arbitrary $\varepsilon > 0$.

(ii) By Theorem 7.2 there is at most countably many local extrema. I.e., a function with the above properties can only have countably many values. Any non-constant continuous function has $2^{\aleph_0} > \aleph_0$ values. (This is special case of Exercise 7.B.)

(iii) Suppose that $f\colon [a,b] \to \mathbb{R}$ is a function such that it has a local maximum at each point of $[a,b]$ but it is not constant on any interval.

By induction we can construct a sequence of points x_i and closed intervals with the properties:

- x_i is a local maximum on I_i;
- $f(x_{i+1}) < f(x_i)$;
- $I_{i+1} \subseteq I_i$;
- $\operatorname{diam}(I_i) \searrow 0$.

Inductive step: Since f is not constant on I_i, there is a point $x_{i+1} \in I_i$ such that $f(x_{i+1}) < f(x_i)$. Since f has a local maximum at x_{i+1}, there is a neighborhood of x_{i+1} on which the values are at most $f(x_{i+1})$. We can take a smaller closed interval I_{i+1} in this neighborhood such that, at the same time, $I_{i+1} \subseteq I_i$ and $\operatorname{diam}(I_i) \leq 2^{-i}$.

By Cantor Intersection Theorem there is a unique point $x \in \bigcap_{i\in\mathbb{N}} I_i$. Clearly, $\lim_{i\to\infty} x_i = x$. We also have $f(x) < f(x_i)$ for each $i \in \mathbb{N}$ (since $x \leq f(x_{i+1}) < f(x_i)$). So x is not a local maximum, which is a contradiction.

(iv) $f(0) = 1$ and $f(x) = 1 - \frac{1}{n}$ for $x \in (\frac{1}{n+1}, \frac{1}{n}]$. □

Exercise 10.T If a function is semicontinuous and Darboux continuous, is it necessarily continuous?

Solution. Let us define

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0, \\ -1 & x = 0. \end{cases}$$

The function f is lower semicontinuous, Darboux, but it is not continuous at the point 0. □

Exercise 10.U Let $f \in \mathscr{C}^+$ be bounded. Show that the set $\{x : f^{\downarrow}(x) \neq f(x)\}$ is meagre. Give a bounded function g for which $\{x : g^{\downarrow}(x) \neq g^{\uparrow}(x)\}$ is not meagre.

Exercise 10.V Are the following two statements true?
(i) If $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and if $g\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) := \sup\{f(t); x - 1 < t < x + 1\}$$

then g is lower semicontinuous.
(ii) If $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and if $h\colon \mathbb{R} \to \mathbb{R}$ is defined by then h is lower semicontinuous.

Solution. (i) By contradiction. Suppose that g is not lower semicontinuous. This implies that there is a sequence $x_n \to x$ such that $\lim_{n\to\infty} g(x_n) < g(x)$. We can get, moreover, that this sequence is monotone, w.l.o.g. let us assume that $x_n \to x$. Hence we have $\sup\{g(t); x_n - 1 < t < x_n + 1\} < g(x) - \varepsilon$ for some $\varepsilon > 0$. This implies that the values of g on the interval $(x_n - 1, x + 1)$ are smaller than $g(x) - \varepsilon$. Since $x_n \to x$, we get $(x - 1, x + 1) = \bigcup_{n=1}^{\infty}$ and this implies that $\sup\{g(t); t \in (x - 1, x + 1)\} < g(x) - \varepsilon$, a contradiction.

(ii) Note that if $f = \chi_A$, then $g = \chi_{B(A,1)}$ is the characteristic function of the open unit ball around A. Similarly, $h = \chi_{\overline{B}(A,1)}$ is the characteristic function of the closed unit ball around A.

For example for $A = 3\mathbb{Z}$ we get $\overline{B}(A, 1) = \bigcup_{x\in\mathbb{Z}}[x - 1, x + 1]$, which is not an open set. Hence for $f = \chi_A$ the function h is not lower semicontinuous. (See Theorem 10.2(ii).) □

Observation. $f^{\downarrow}(x) = \lim_{n\to\infty} \sup\{g(t); x-1/n < t < x+1/n\} = \inf_{n\in\mathbb{N}} \sup\{g(t); x-1/n < t < x+1/n\}$

So in the proof of (i) we have expressed $f^{\downarrow}$ as limit (infimum) of a non-increasing sequence of lower semicontinuous functions. $\square$

3.11 Functions of the first class of Baire

Exercise 11.A Show the following: A function of the first class need not be semicontinuous. A function of the first class need not be a derivative. A function of the first class need not be of bounded variation.

Exercise 11.B Show that a function $[a, b] \to \mathbb{R}$ with only finitely many discontinuities is of the first class (see Theorem 11.8).

If $t_1 \dots t_n$ are discontinuities, we can define continuous function, which agrees with the original function outside $\bigcup_{i=1}^{n} (t_i - \varepsilon, t_i + \varepsilon)$.

Exercise 11.C Let f be a differentiable function on an interval. Show that f' is of the first class. (Be careful!)

We have already seen that for functions on real line $f'(x) = \lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n}$ is a pointwise limit of a sequence of continuous function. If we are working with functions defined on some interval I, the problem with the above argument is that $f(x + 1/n)$ might be undefined.

So let $f\colon I \to \mathbb{R}$, where I is a non-degenerate interval, and let x_0 be an interior point of this interval. Then we can define a sequence of continuous functions such that

$$g_n(x) = \begin{cases} \frac{f(x+1/n)-f(x)}{1/n} & x \in I \setminus (x_0 - 1/n, x_0 + 1/n), x < x_0 \\ \frac{f(x)-f(x-1/n)}{1/n} & x \in I \setminus (x_0 - 1/n, x_0 + 1/n), x > x_0 \\ f'(x_0) & x = x_0 \end{cases}$$

and for $x \in (x_0 - 1/n, x_0 + 1/n) \setminus \{x_0\}$ we define $f_n(x)$ as an arbitrary continuous extension of the above values.

It is easy to see that g_n converges to f' pointwise.[20]

Exercise 11.F Let $f\colon [0, 1] \to \mathbb{R}$ be such that its restriction to $(0, 1]$ is of the first class. Show that f itself is also of the first class.

We can choose continuous functions f_n such that f_n and f have the same values on the set $\{0\} \cup (1/n, 1]$.

[20] My question: Can we do something similar for the symmetric derivative using the fraction $\frac{f(x+1/n)-f(x-1/n)}{2/n}$?

Theorem 11.4. The fact that the set of continuity points of f is dense, is omitted. It is only shown that discontinuity points form a F_σ meager set.

Proof. By Exercise 6.I every F_σ meager set has empty interior. Therefore the complement is dense. $\square$

Exercise 11.G. Let I be an interval, let $f\colon I \to \mathbb{R}$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \begin{cases} f(x) & \text{if } x \in I, \\ 0 & \text{if } x \notin I. \end{cases}$$

Show that $f \in \mathscr{B}^1(I)$ if and only if $g \in \mathscr{B}^1(\mathbb{R})$.

Exercise 11.H Let $A \subseteq \mathbb{R}$ be a meagre F_σ. Show that there exists a (semi-continuous) function $f\colon \mathbb{R} \to \mathbb{R}$ of the first class such that the elements of A are just the points of discontinuity of f. (Choose closed sets $K_1 \subseteq K_2 \subseteq \dots$ whose union is A, and set $f := \sum\limits_{n=1}^{\infty} 2^{-n}\xi_{K_n}$.)

Proof. Any F_σ meagre set is a union of increasing sequence of closed boundary sets (see Exercise 6.I).

If was already shown that the continuity points of the function of the form $\sum\limits_{k=1}^{\infty} 2^{-k}\xi_{K_k}$ are precisely the points from the set $\bigcup\limits_{k=1}^{\infty} K_k$ (Exercise 7.G).

It is relatively easy to see that every partial sum $\sum\limits_{k=1}^{n} 2^{-k}\xi_{K_k}$ is a lower semi-continuous function. (Preimage of each $[a, \infty)$ is a union of finitely many closed sets.) Hence the function f is a supremum of lower semicontinuous functions; so it is again lower semicontinuous. By Theorem 10.6 every lower semicontinuous function belongs to $\mathcal{B}_1$.

Other possibility would be using the fact that $\mathcal{B}_1$ is closed under limits of uniformly convergent sequences. (This fact is stated as a theorem in the book only after this exercise.) $\square$

Proof of Theorem 11.7. The following observation can be isolated as a part of the proof of Theorem 11.7.

Lemma. *Let $(s_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ be a double sequence with the following properties:*

(i) for each $n \in \mathbb{N}$ the limit

$$a_n = \lim_{k\to\infty} s_{nk}$$

exists;

(ii) there exists a real sequence (M_n) such that $\sum\limits_{n=1}^{\infty} M_n < +\infty$ and

$$(\forall k \in \mathbb{N})|s_{nk}| \le M_n. \tag{3.1}$$

Then

$$\lim_{k\to\infty}\sum_{n=1}^{\infty}|s_{nk}-a_n|=0 \tag{3.2}$$

As a consequence we get

$$\sum_{n=1}^{\infty}a_n=\lim_{k\to\infty}\sum_{n=1}^{\infty}s_{nk}. \tag{3.3}$$

In the other words,

$$\sum_{n=1}^{\infty}\lim_{k\to\infty}s_{nk}=\lim_{k\to\infty}\sum_{n=1}^{\infty}s_{nk}.$$

Note that (3.1) implies $|a_n|\le M_n$ and thus the series $\sum\limits_{n=1}^{\infty}a_n$ and each $\sum\limits_{n=1}^{\infty}s_{nk}$ is absolutely convergent.

The above result can be considered a special case of Lebesgue dominated convergence theorem (for counting measure on $\mathbb{N}$), but we will provide a proof anyway.

Proof. Let $\varepsilon>0$.

Since $\sum\limits_{n+1}^{\infty}M_n<\infty$, there is an $N\in\mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty}M_n<\frac{\varepsilon}{3}.$$

Thus, for each $k\in\mathbb{N}$, we have

$$\sum_{n=N+1}^{\infty}|s_{nk}-a_n|\le\sum_{n=N+1}^{\infty}|s_{nk}|+\sum_{n=N+1}^{\infty}|a_n|<\frac{2\varepsilon}{3}.$$

Since $a_n=\lim\limits_{k\to\infty}s_{nk}$, we can choose k_0 such that for $k\ge k_0$ and for $n\in\{1,2,\dots,N\}$

$$\begin{gathered}|s_{1k}-a_1|<\frac{\varepsilon}{3}\cdot\frac{1}{2}\\ \vdots\\ |s_{Nk}-a_N|<\frac{\varepsilon}{3}\cdot\frac{1}{2^N}\end{gathered}$$

which implies

$$(\forall k\ge k_0)\sum_{n=1}^{N}|s_{nk}-a_n|<\frac{\varepsilon}{3}.$$

Together we get

$$(\forall k \geq k_0) \sum_{n=1}^{\infty} |s_{nk} - a_n| < \varepsilon.$$

So we have shown (3.2):

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} |s_{nk} - a_n| = 0.$$

To see that (3.2) implies (3.3) just observe that

$$0 \leq \left| \sum_{n=1}^{N} s_{nk} - \sum_{n=1}^{N} a_n \right| \leq \sum_{n=1}^{N} |s_{nk} - a_n|.$$

Taking $N \to \infty$ yields

$$0 \leq \left| \sum_{n=1}^{\infty} s_{nk} - \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |s_{nk} - a_n|.$$

Taking $k \to \infty$ yields

$$\lim_{k\to\infty} \left| \sum_{n=1}^{\infty} s_{nk} - \sum_{n=1}^{\infty} a_n \right| = 0$$

which is equivalent to (3.3):

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} s_{nk} = \sum_{n=1}^{\infty} a_n.$$

□

Using this observation we can summarize the proof of Theorem 11.7 as follows.

Proof. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ converges uniformly and each f_n belongs to $\mathscr{B}^1(X)$. Then we can choose a subsequence g_n such that $\|f(x) - g_n(x)\|_\infty \leq 1/2^{n+1}$ for each $n \in \mathbb{N}$.

If we put $h_n := g_{n+1} - g_n$ and $h_1 = g_1$ then each h_n belongs to $\mathscr{B}^1(X)$ and fulfills

$$\|h_n\| \leq \frac{1}{2^n}.$$

We want to show that $\sum_{n=1}^{\infty} h_n \in \mathscr{B}^1$.

For each h_n there exists a sequence $(s_{nk})_{k=1}^{\infty}$ of continuous functions which converges to h_n pointwise, i.e.,

$$h_n(x) = \lim_{k\to\infty} s_{nk}(x)$$

for each $x \in X$.

Since $\|h_n\|_\infty \leq 2^{-n}$, we may assume (according to Exercise 11.D) that $\|s_{nk}\|_\infty \leq 2^{-n}$ for each k. This implies that the series $\sum\limits_{n=1}^{\infty} s_{nk}(x)$ is uniformly convergent and the limit $t_k = \lim\limits_{k\to\infty} s_{nk}$ is a continuous functions for each k.

From Lemma 3.11 we have, for each $x \in X$,

$$\lim_{k\to\infty} t_k(x) = \lim_{k\to\infty} \sum_{n=1}^{\infty} s_{nk}(x) = \sum_{n=1}^{\infty} \lim_{k\to\infty} s_{nk}(x) = \sum_{n=1}^{\infty} h_n(x).$$

We have obtained that t_k converges pointwise to $\sum_n h_n$, which means that $\sum_n h_n \in \mathscr{B}^1(X)$. □

Exercise 11.N. TODO [21]

Exercise 11.P. (i) If $f\colon \mathbb{R} \to \mathbb{R}$ has a closed graph, then the points of discontinuity of f form a closed set with empty interior.
(ii) Conversely, if A is a closed subset of $\mathbb{R}$ with empty interior, then there exists an $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is closed and such that A is just the set of all points of discontinuity of f (Hint. Let ϕ be as in Exercise 1.P. Define $f(x) := 0$ for $x \in A$, $f(x) := \phi(x)^{-1}$ for $x \in \mathbb{R} \setminus A$.)

Solution of (i). From Theorem 11.10 we get that $f \in \mathscr{B}^1$. By Theorem 11.4 the set of all discontinuity points is a meager F_σ set. Thus from Exercise 6.I we get that it is has empty interior. It only remains to show that this set is closed.

The set C_f of all continuity points is open. If $x_0 \in C_f$, then for each $\varepsilon > 0$ there is $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. Hence f is bounded on the interval $(x_0 - \delta, x_0 + \delta)$.

Let $\widehat{x} \in (x_0 - \delta, x_0 + \delta)$. Suppose that $\widehat{x} \notin D_f$. This implies that

$$a := \liminf_{x\to\widehat{x}} f(x) < \limsup_{x\to\widehat{x}} f(x) =: b.$$

(The boundedness of f on this interval implies that a, b are finite numbers.) Since the graph Γ_f is closed, we get that the two points $[\widehat{x}, a]$ and $[\widehat{x}, b]$ both belong to Γ_f. This is a contradiction.

So we have shown that $(x_0 - \delta, x_0 + \delta) \subseteq C_f$.

Since each point $x_0 \in C_f$ has a neighborhood which lies entirely in C_f, the set C_f is open. □

[21] `http://mathoverflow.net/questions/117837/uniform-closure-of-subspaces-of-baire-class-1`; Note that such an example would contradict the (presumably erroneous) claim in the otherwise nice text "A second course on real functions" by van Rooij and Schikhof, Cambridge (1982), Exercise 11N, p. 69. The point of the Exercise is to prove all the higher Baire classes are uniformly complete, which is true, but the hint claims that $B_1(A)$ is always uniformly complete for any linear space A.

Solution of (ii). Let A be a closed set with empty interior. Let us define

$$f(x) = \begin{cases} 0 & x \in A, \\ \frac{1}{d(x,A)} & x \notin A. \end{cases}$$

A is the set of discontinuity points of f. If $x \notin A$, then on some neighborhood of x this function coincides with the continuous function $1/d(x, A)$, and thus x is a continuity point.

If $x \in A$, then each neighborhood of x contains some point from $\mathbb{R} \setminus A$ (since $\operatorname{Int} A = \emptyset$). From these points we can contruct a sequence x_n such that x_n to x and $f(x_n) \to \infty$. Thus f is not continuous at x.

The graph Γ_f is a closed set. We want to show that if $x_n \to x$ and $f(x_n) \to a$ (where $a \in \mathbb{R}$), then $a = f(x)$.

If $x \notin A$ then, starting from some n_0, we have $x_n \notin A$. Since f is continuous on the set $X \setminus A$, we get $f(x_n) \to f(x)$.

If $x \in A$, we will consider two possibilities. First, suppose that there is some n_0 such that for each $n \geq n_0$ we have $x_n \in A$. This implies that $f(x_n) = 0$ for $n \geq n_0$, i.e. starting from n_0 the sequence $f(x_n)$ is constant and $\lim\limits_{n\to\infty} f(x_n) = 0 = f(x)$.

The remaining possibility is that there are infinitely many n's with $x_n \notin A$. In this case we can choose a subsequence such that $x_{n_k} \to x$ and $f(x_{n_k}) = \frac{1}{d(x_{n_k},A)} \to \infty$. So in this case $f(x_n)$ cannot converge to a real number a. □

Exercise 11.Q. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that for every integer n the restriction of f to $[n, n+1]$ is of the first class. Then f itself is of the first class.

Solution. We restrict the continuous functions f_k, which converge pointwise to f, to the intervals $[n + 1/k, n + 1 - 1/k]$ and add piecewise linear functions joining the points $[f(x), x]$ for $x = n - 1/k$, $x = n$ and $x = n + 1/k$. □

Solution. Each $f^{-1}(U) \cap [n, n+1]$ is F_σ. □

Exercise 11.R. Let $f \in \mathscr{B}^1(\mathbb{R})$ be such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that $\{x; f(x) \neq 0\}$ is meagre.

Solution. By Theorem 11.4 the set of all discontinuity points of f is a meager F_σ-set. All points from the set $\{x; f(x) \neq 0\}$ are discontinuity points of f. □

Exercise 11.S. Let $f \in \mathscr{B}^1(\mathbb{R})$ and let g be a continuous function on the set $f[\mathbb{R}]$. Show that $g \circ f \in \mathscr{B}^1(\mathbb{R})$. (Warning. If $f_1, f_2, \ldots$ are continuous functions and if $\lim\limits_{n\to\infty} f_n = f$ then g may not be defined on $f_n(\mathbb{R})$!)

Solution. $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

Since g is continuous, the set $g^{-1}(U)$ is open in the relative topology on $f[\mathbb{R}]$. This means that $g^{-1}(U) = V \cap f[\mathbb{R}]$ for some open subset V of $\mathbb{R}$. We have $f^{-1}(V) = g^{-1}(U)$, and thus $(g \circ f)^{-1}(U)$ is F_σ. □

Exercise 11.T. Let I be an interval, let $f\colon I \to \mathbb{R}$. For $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} -n & \text{if } x \in I \text{ and } f(x) \le -n, \\ f(x) & \text{if } x \in I \text{ and } -n < f(x) < n, \\ n & \text{if } x \in I \text{ and } f(x) \ge n. \end{cases}$$

If each f_n is of the first class, then so is f.

Solution. Notice that $f^{-1}((a,b)) = f_n^{-1}((a,b))$ for sufficiently large n. □

Exercise 11.U (i) Let $X \subseteq \mathbb{R}$ be countable. Then every function $X \to \mathbb{R}$ is an element of $\mathscr{B}^1(X)$. (Hint. For $x_1, \dots, x_n \in X$ ($x_i \ne x_j$ when $i \ne j$) and $a_1, \dots, a_n \in \mathbb{R}$ there is a continuous function $f\colon X \to \mathbb{R}$ such that $f(x) = \alpha_i$ for each i.)
(ii) The following statement is false. For every countable set $X \subseteq \mathbb{R}$ and every $f \in \mathscr{B}^1(X)$ there exists an $f \in \mathscr{B}^1(\mathbb{R})$ such that g is the restriction of f to X. (Let $X := \mathbb{Q} \cup (\mathbb{Q} + \pi)$, $g(x) := 0$ if $x \in \mathbb{Q}$, $g(x) := 1$ if $x \in \mathbb{Q} + \pi$.)

Solution of (i). TODO

□

Solution of (ii). We have $X = \mathbb{Q} \cup (\mathbb{Q} + \pi)$ and

$$g(x) = \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \in \mathbb{Q} + \pi. \end{cases}$$

If $f\colon \mathbb{R} \to \mathbb{R}$ and $f|_X = g$ then every real number is a discontinuity point of f. This contradicts Theorem 11.4. □

Would it be possible to show that there is such function using cardinality argument? No, since the cardinalities are the same.

There are $\mathfrak{c}$ continuous functions from $\mathbb{R}$ to $\mathbb{R}$. The cardinality of all pointwise limits of such functions is at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

If we fix a countable subset $X \subseteq \mathbb{R}$, then there is $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ functions from X to $\mathbb{R}$. There is $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ different countable subsets of $\mathbb{R}$. TODO

Exercise 11.V. For $n \in \mathbb{N}$ let $S_n := \{t2^{-n} : t \text{ is an odd integer}\}$. Observe that the sets $S_1, S_2, \dots$ are pairwise disjoint and that for every infinite subset T of $\mathbb{N}$, $\bigcup_{n \in T} S_n$ is dense in $\mathbb{R}$.

Let $b, a_1, a_2, \dots$ be real numbers. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} a_n & \text{if } n \in \mathbb{N}, x \in S_n, \\ b & \text{if } x \in \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} S_n. \end{cases}$$

Show that $f \in \mathscr{B}^1(\mathbb{R})$ if and only if $\lim_{n \to \infty} a_n = b$.

Solution. $\boxed{\Rightarrow}$ If $\lim a_n \neq b$, then every real number is a discontinuity point of f.

First we show this $x \notin \bigcup_n S_n$. Every neighborhood of x contains points belonging all but finitely many S_n's. Choosing one such point (the closest one to x) for each n we obtain a sequence such that $x_n \to x$ and $f(x_n) \not\to f(x)$.

Now let $x \in S_n$. If $a_n = b$, then we can argue as in the preceding case. If $a_n \neq b$ then we can use the fact that every neighborhood of x contains a point from $\mathbb{R} \setminus \bigcup_n S_n$.

The fact that the set of discontinuity points is $\mathbb{R}$ contradicts Theorem 11.4 $\square$

Exercise 11.X. Let $\psi\colon \mathbb{R} \to \mathbb{R}$. Show that the following conditions are equivalent.
(α) For every $f \in \mathscr{B}^1(\mathbb{R})$ we have $f \circ \psi \in \mathscr{B}^1(\mathbb{R})$.
(β) For every F_σ-set $X \subseteq \mathbb{R}$, $\psi^{-1}(X)$ is an F_σ.
(γ) For every open $U \subseteq \mathbb{R}$, $\psi^{-1}(U)$ is both an F_σ and a G_δ.

Proof. $(\alpha) \Rightarrow \gamma$ Let U be an open set and $f = \chi_U$. Since every open set is F_σ and G_δ, by Theorem 11.6 $f \in \mathscr{B}^1(X)$. Notice that

$$\chi_U \circ \psi(x) = \begin{Bmatrix} 1 & x \in U \\ 0 & x \notin U \end{Bmatrix} = \chi_{\psi^{-1}(U)}$$

Again from Theorem 11.6 we get that $\psi^{-1}(()U)$ is both F_σ and G_δ.

$(\gamma) \Rightarrow (\beta)$ By taking the complements we get that preimage of every closed set is F_σ (and G_δ). Hence if $F = \bigcup_n C_n$ is a countable union of closed sets, then $f^{-1}(F) = \bigcup_n f^{-1}(C_n)$ is a countable union of F_σ sets, which is again F_σ.

$(\beta) \Rightarrow (\alpha)$ $f \circ \psi^{-1}(U) = \psi^{-1}(f^{-1}(U))$ is a preimage of an F_σ set $f^{-1}(U)$, so by (β), it is again an F_σ-set. $\square$

Exercise 11.Y. Let $f\colon \mathbb{R} \to \mathbb{R}$, let Γ_f be the graph of f.
(i) Show that

$$\Gamma_f = \bigcap_{\substack{a,b \in \mathbb{Q} \\ a<b}} \{(x,y) \in \mathbb{R}^2; a \leq f(x) \leq b \text{ or } y > b \text{ or } y < a\}$$

Deduce that, if $f \in \mathscr{B}^1$, then Γ_f is a G_δ subset of $\mathbb{R}^2$.

Solution. (i) It suffices to show that the set $\{(x,y) \in \mathbb{R}^2; a \leq f(x) \leq b\}$ is G_δ. $\square$

(ii) Let $\{q_1, q_2, \dots\}$ be an enumeration of $\mathbb{Q}$. The following formula defines an $f\colon \mathbb{R} \to \mathbb{R}$ that is not of the first class although its graph is a G_δ.

$$\begin{cases} f(q_n) = n^{-1} & \text{for all } n \in \mathbb{N}, \\ f(x) = -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Solution. (ii) The graph of the function consists of two disjoint parts

$$A = (\mathbb{R} \setminus \mathbb{Q}) \times \{-1\}$$
$$B = \{(q_n, 1/n); n \in \mathbb{N}\}.$$

We will show that each of these sets can be expressed as a countable intersection of open sets; $A = \bigcap_n U_n$, $B = \bigcap_n V_n$. Since these sets will have an addition property that $U_n \cap V_k = \emptyset$, it will be obvious that then $\Gamma_f = A \cup B$ is a G_δ-set, too. (And we already know from Theorem 6.2 that union of two G_δ-sets is G_δ.)

We define $U_n = \mathbb{R} \setminus \{q_1, \dots, q_n\} \times (-1-\varepsilon_n, -1+\varepsilon_n)$, where $\varepsilon_n \searrow 0$, $\varepsilon_n < 1$. Clearly $A = \bigcap_n U_n$.

Let us denote $P_i = (q_i, \frac{1}{i})$; i.e., we have $B = \{P_i; i \in \mathbb{N}\}$.

For the second part we choose $\varepsilon_n \to 0$ in a such way that $\varepsilon_n < \frac{1}{n(n+1)}$.

Notice that for any $j, k \in \{1, 2, \dots, n\}$, $j < k$, we have

$$\frac{1}{j} - \frac{1}{k} \geq \frac{1}{j} - \frac{1}{j+1} = \frac{1}{j(j+1)} \geq \frac{1}{n(n+1)} > \varepsilon_n.$$

Since $d_2(P_j, P_k) > |1/j - 1/k|$, we see that the ball $B(P_n, \varepsilon_n)$ contains only one of the points $(q_k, 1/k)$, $k \in \{1, 2, \dots, n\}$.

For $k > n$ we have $d_2(P_n, P_k) > \frac{1}{n} - \frac{1}{k} > \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > \varepsilon_n$. So we get that the open ball $B(P_n, \varepsilon_n)$ contains only one point of the set B.

Now we define $V_n = \mathbb{R} \times (0, \frac{1}{n}) \cup (\bigcup_{k=1}^{n} B(P_k, \varepsilon_n))$. We want to show that $B = \bigcap_{n=1}^{\infty} V_n$. It is clear that $B \subseteq \bigcap_n V_n$.

Now let $P = (x, y) \notin B$. If $y \leq 0$ or $y > 1$, then $y \notin V_n$ for each n. If $y \in (0, 1)$ we can choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < y \leq \frac{1}{n}$. If $y \neq \frac{1}{n}$, then we have $y \notin V_{n+1}$.

So the only remaining case is that $P = (x, 1/n)$ where $x \neq q_n$. Using the same argument as above, we get $d_2(P, P_k) > \varepsilon_n$ for each k, hence $y \notin B(q_k, \varepsilon_n)$ for $k \neq n$. Since $x \neq q_n$, by choosing m large enough we get $y \notin B(P_n, \varepsilon_m)$. For such m we have $P \notin V_m$. □

Challenge 11.Z. Let $f\colon \mathbb{R} \to \mathbb{R}$, let Γ_f be the graph of f. If Γ_f is an F_σ subset of $\mathbb{R}^2$, then Γ_f is also G_δ subset of $\mathbb{R}^2$.

Proposition. *If Y is a compact Hausdorff space and X is a topological space, then the projection $p_X\colon X \times Y \to X$ is closed.*

If fact, the above result is a characterization of compact spaces in the class of Hausdorff spaces, this is known as Kuratowski theorem, see [E, Theorem 3.1.16].

Proof. Let C be a closed subset of $X \times Y$. We will show that $p_X[C]$ is closed, too.

Let $(x_d)_{d\in D}$ be a net such that each x_d belongs to $p_X[C]$ and $x_d \to x$. We want to show that $x \in C$.

The condition $x_d \in p_X[C]$ implies the existence of y_d's such that (x_d, y_d) in C. Since Y is compact, there is a convergent subnet $(y_{d'})_{d'\in D'}$ of $(y_d)_{d\in D}$. For this subnet we have $x_{d'} \to x$, $y_{d'} \to y$ and closedness of C implies $(x, y) \in C$. Therefore $x \in p_X[C]$. □

Lemma. *Let Y be a compact Hausdorff space and $f\colon X \to Y$. If Γ_f is closed subset of $X \times Y$, then f is continuous.*

In fact, the condition in the above lemma are equivalent, see [AB, Theorem 2.58][22]

Proof. Since Y is compact, the map $p_X\colon X \times Y \to X$ is closed. Hence for any closed subset C of Y we get that

$$f^{-1}(C) = p_X[X \times C \cap \Gamma_f]$$

is closed. □

Proof of Challenge 11.Z. Suppose that Γ_f is F_σ. This means that $\Gamma_f = \bigcup C_n$ for some closed sets C_n. W.l.o.g. we can assume that the sequence C_n is non-decreasing and that $C_n \subseteq \mathbb{R} \times [-n, n]$. (If A_n is an arbitrary system of closed sets such that $\Gamma_f = \bigcup A_n$, then by putting $B_n := \bigcup\limits_{k=1}^{n} A_k$ we get an increasing system. If we put $C_n := B_n \cap \mathbb{R} \times [-n, n]$, we get a system with the required properties.)

Let us denote $X_n = p_1[C_n]$. By the above proposition the set X_n is closed.

The closed set C_n is the graph of the restriction $f|_{X_n}\colon X_n \to [-n, n]$. The space $[-n, n]$ is compact, so by the above lemma the map $f|_{X_n}$ is continuous.

Since X_n is closed, by Tietze's extension theorem we get that there is a continuous extension $g_n\colon \mathbb{R} \to [-n, n]$.

The fact that $\bigcup X_n = \mathbb{R}$ implies that g_n converges pointwise to f. Hence $f \in \mathscr{B}^1$ and, according to Exercise 11.Y, Γ_f is G_δ. □

3.12 Riemann integrable functions

Exercise 12.F. Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable. If $f = 0$ a.e., then $\int_a^b f(x)\,\mathrm{d}x = 0$. (Compare Exercise 4.C(iv).) More than that: if $f = 0$ on a dense set, then $\int_a^b f(x)\,\mathrm{d}x = 0$.

Exercise 12.O. Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable. Consider the function $f\colon x \mapsto \int_a^x f(t)\,\mathrm{d}t$. Prove that $F'(x) = f(x)$ for every point of continuity x of f. In particular, F is differentiable a.e. on $[a, b]$ and $F' = f$ a.e. on $[a, b]$.

[22] or `http://at.yorku.ca/cgi-bin/bbqa?forum=ask_a_topologist_2004;task=show_msg;msg=0625.0001`.

4 Differentiation

4.13 Differentiable functions

$\mathcal{D}$ =all differentiable functions on interval I

Exercise 13.A. Show that $\mathcal{D}$ is not closed with respect to sup, inf and uniform limits.

$f(x) = |x|$ is not differentiable. It is equal to $\liminf f_n(x) = \inf f_n(x)$ for

$$f_n(x) = \begin{cases} \frac{1}{n}\left(\frac{1}{2} + \frac{x^2}{2}\right) & |x| < \frac{1}{n}, \\ |x| & |x| \geq \frac{1}{n}. \end{cases}$$

Exercise 13.B. Let $f \in \mathcal{D}$. Show that f' is continuous if and only if the function $\Phi_1 f$, defined via

$$\Phi_1 f(x, y) := \frac{f(x) - f(y)}{x - y}, \qquad (x, y \in I, x \neq y)$$

can be extended to a continuous function on $I \times I$.

Solution. By definition $f'(x) = \lim_{t \to 0} \Phi_1 f(x + t, x)$.

$\boxed{\Leftarrow}$ If there exists a continuous extension g of the function $\Phi_1 f$, then necessarily

$$g(x, x) = f'(x).$$

Then the continuity of g implies continuity of $f'(x) = g(x, x)$.

$\boxed{\Rightarrow}$ We define

$$g(x, y) = \begin{cases} \Phi_1 f(x, y) & x \neq y, \\ f'(x) & x = y. \end{cases}$$

Then the function g is continuous.

It is clear that g is continuous at any point (x, y) such that $x \neq y$.

It is also clear that $g(x, x)$ is close to $g(x_0, x_0)$ if x is close to x_0, by the continuity of $f'(x) = g(x, x)$.

Now let $\delta > 0$ be such that $|f'(\xi) - f'(x_0)| < \varepsilon$ whenever $|\xi - x_0| < \delta$. If $(x, y) \in I \times I$ is a point such that $x \neq y$, $|x - x_0| < \delta$ and $|y - x_0| < \delta$ and $x \neq y$. Then by intermediate value theorem there is a point ξ between x and y such that $\Phi_1 f(x, y) = f'(\xi)$. Since ξ is between x and y, we have $|\xi - x_0| < \delta$ and $|\Phi_1 f(x, y) - g(x_0, x_0)| = |f'(\xi) - f'(x_0)| < \varepsilon$

$\square$

Exercise 13.C. Let X be a nonempty subset of $\mathbb{R}$ without isolated points. For an $f \colon X \to \mathbb{R}$ we define the notions 'continuous', 'differentiable', 'increasing at $x \in X$, 'monotone' in the natural way.

Now let $f \colon \mathbb{Q} \to \mathbb{R}$.

(i) Let f be increasing at every point of $\mathbb{Q}$. Is f increasing?
(ii) Let f be differentiable. Is f continuous?
(iii) Let f be increasing and differentiable. Is $f' \geq 0$?
(iv) Let f be differentiable and $f' \geq 0$. Is f increasing?
(v) Let f be differentiable. Is there a ξ between 0 and 1 such that $f(1)-f(0) = f'(\xi)$?
(vi) Let f be monotone. Is f somewhere differentiable?

Try to answer similar questions for $f\colon \mathbb{D} \to \mathbb{R}$.

Solution. (i) No. Let us define $f\colon \mathbb{Q} \to \mathbb{R}$

$$f(x) = \begin{cases} x+1 & x < \sqrt{2}, \\ x & x > \sqrt{2}. \end{cases}$$

This function is increasing at every point of $\mathbb{Q}$, but not increasing.

(ii) Yes. Let $\varepsilon > 0$. If $\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < \varepsilon$ then

$$|f(y) - f(x)| < |f'(x)||x-y| + \varepsilon.$$

If we choose δ such that $|f'(x)|\delta < \varepsilon$, then we get $|f(y) - f(x)| < 2\varepsilon$ whenever $|y - x| < \delta$.

(iii) Yes. $f'(x) = \lim\limits_{y \to x} \frac{f(y)-f(x)}{y-x} \geq 0$ since $\frac{f(y)-f(x)}{y-x}$ for any $x, y \in \mathbb{Q}$.

(iv) No. For example

$$f(x) = \begin{cases} x & x < \sqrt{2}, \\ x-1 & x > \sqrt{2}. \end{cases}$$

(v) No. Let us define $f\colon \mathbb{Q} \to \mathbb{R}$

$$f(x) = \begin{cases} x+1 & x < \frac{\sqrt{2}}{2}, \\ x & x > \frac{\sqrt{2}}{2}. \end{cases}$$

Then $f(2) - f(0) = 2$, but $f'(x) = 1$ for each $x \in \mathbb{Q}$.

(vi) Let $\{q_n; n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Let $f\colon \mathbb{Q} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{q_k < x} \frac{1}{k^2}.$$

This function is not continuous and thus not differentiable.

To see that it is not continuous at $x = q_n \in \mathbb{Q}$ it suffices to notice that for $y < x < z$ we have $f(z) - f(x) > \frac{1}{n^2}$, and thus

$$\lim_{y \to x^-} f(x) \neq \lim_{z \to x^+} f(x).$$

TODO $\mathbb{D}$ $\square$

We may notice the following estimate, which will be useful in the following two exercises: $\sum_{k>N} 3^{-k!} \leq 3^{-N!} \sum_{k>N} 3^{k-N} = \frac{1}{2} 3^{-N!}$, hence

$$\sum_{k>N} 3^{-k!} \leq \frac{1}{2} 3^{-N!} < 3^{-N!}.$$

Exercise 13.D. By Exercise 4.D, $\mathbb{D}$ is the set of all numbers $\sum_{n=1}^{\infty} a_n 3^{-n}$ where $a_n \in \{0, 2\}$. Define a function $f\colon \mathbb{D} \to \mathbb{R}$ as follows.

$$f\left(\sum_{n=1}^{\infty} a_n 3^{-n}\right) := \sum_{n=1}^{\infty} a_n 3^{-n!}.$$

(i) Show that f is injective by proving that, if $x, y \in \mathbb{D}$, $N \in \mathbb{N}$, then

$$\text{if } y - x > 3^{-N}, \text{ then } f(y) - f(x) \geq 3^{-N!}$$

(ii) For each $m \in \mathbb{N}$ there exists a $c_m \in \mathbb{R}$ with

$$|f(x) - f(y)| \leq c_m |y - x|^m \qquad (x, y \in \mathbb{D})$$

(Thus, f is differentiable and $f' = 0$, f satisfies Lipschitz condition of all positive orders.)

(iii) For every $m \in \mathbb{N}$ we have

$$\lim_{\substack{x \in \mathbb{D} \\ x \to y}} \frac{f(x) - f(y)}{(x - y)^m} = 0 \qquad (y \in \mathbb{D})$$

Solution. (i) If $y = \sum b_n 3^{-n}$, $x = \sum a_n 3^{-n}$ and $y - x > 3^{-N}$, then the first position n_0 such that $b_{n_0} > a_{n_0}$ is $n_0 \leq N$. [23] Therefore

$$f(y) - f(x) \geq 2 \cdot 3^{-N!} - 2 \sum_{k > N} 3^{-k!} \geq 2 \cdot 3^{-N!} - 3^{-N!} = 3^{-N!}.$$

(ii) Suppose that $y > x$, $y = \sum b_n 3^{-n}$ and $x = \sum a_n 3^{-n}$. If n_0 is the first position where $b_{n_0} > a_{n_0}$ then

$$y - x \geq 3^{-n_0}$$

and

$$f(y) - f(x) \leq 2 \sum_{k \geq n_0} 3^{k!} \leq 2 \cdot 3^{(n_0 - 1)!}.$$

So it suffices to find a constant c_m such that $2 \cdot 3^{-(n-1)!} \leq c_m (3^{-n})^m$ for each n. But since

$$\frac{2 \cdot 3^{-(n-1)!}}{3^{-nm}} = 2 \cdot 3^{nm - (n-1)!} \to 0,$$

the sequence $\left(\frac{2 \cdot 3^{-(n-1)!}}{3^{-nm}}\right)_{n=1}^{\infty}$ is bounded and there exists such constant c_m.

(iii) If we choose any $n < m$ then we get

$$0 \leq \frac{|f(x) - f(y|}{|x - y|^m} \leq \frac{c_n |x - y|^n}{|x - y|^m} = c_n |x - y|^{n - m}.$$

The RHS converges to 0, so the fraction $\frac{|f(x) - f(y|}{|x - y|^m}$ converges to 0, too. $\square$

[23] $\sum_{k > N} 2 \cdot 3^{-k} = 2 \cdot 3^{-N-1} \frac{1}{1 - \frac{1}{3}} = 3^{-N}$

Exercise 13.E (A weird function on a weird set) Let $X = \{\sum_{n=1}^{\infty} a_n 3^{-n!}; a_1, a_2, \dots \in \{0,1\}\}$ and define $f\colon X \to \mathbb{R}$ by

$$f\left(\sum_{n=1}^{\infty} a_n 3^{-n!}\right) = \sum_{n=1}^{\infty} a_n 27^{-n!}.$$

(i) Show that X is a closed subset of $\mathbb{R}$ without isolated points,
(ii) Show that for every $y \in X$ we have

$$\lim_{\substack{x\in X\\ x\to y}} \frac{f(x)-f(y)}{(x-y)^3} = 1.$$

Solution. (i) *X is homeomorphic to* $\{0,1\}^{\mathbb{N}}$. Let us define for $a = (a_n) \in \{0,1\}^n$

$$h(a) = \sum_{k=1}^{\infty} a_k 3^{-k!}.$$

Notice that if $a = (a_n)$, $b = (b_n)$ are sequences such that $a_n = b_n$ for $n \le N$, then

$$|h(a) - h(b)| \le \sum_{k>N} 3^{k!} \le \sum_{k>N} \frac{3^{-k!}}{2^{k-n}} \le 3^{-N!}.$$

This shows that the map $h\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is continuous (with respect to product topology on $\{0,1\}^{\mathbb{N}}$).

Since it is a continuous map from a compact space $\{0,1\}^{\mathbb{N}}$ to a Hausdorff space X, it is a homeomorphism.

X is closed. The set X is a continuous image of a compact set, so it is compact and, consequently, it is a closed set.

X has no isolated points. The space X is homeomorphic to $\{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$ has no isolated points.

(ii) If $x_n = \sum_{k=1}^{n} a_k^{(n)} 3^{-k!}$ converges to $y = \sum_{k=1}^{n} !b_k 3^{-k!}$ then, starting from some n_0, the expansion of x_n will have the first N "digits" the same as the first N digits of y. (Since the convergence in X is the same as the convergence in $\{0,1\}^{\mathbb{N}}$.)

So let us assume that try to estimate $(f(x) - f(y))/(x-y)$ using the first place k_0 where the difference occurs.

We get:

$$\frac{f(x)-f(y)}{(x-y)^3} = \frac{\sum_{k=k_0}^{\infty} x_k 3^{-3k!}}{\left(\sum_{k=k_0}^{\infty} x_k 3^{-k!}\right)^3} \le \frac{3^{-3k_0!} + 2\cdot 3^{-3(k_0+1)}!}{(3^{-k_0!} - 2\cdot 3^{-(k_0+1)!})^3} =$$

$$= \frac{1 + 2\cdot 3^{-3((k_0+1)!-k!)}}{\left(1 - 2\cdot 3^{-((k_0+1)!-k!)}\right)^3} \overset{k_0\to\infty}{\longrightarrow} 1$$

(In the above expressions of $x-y$ and $f(x)-f(y)$ all x_k's are from $\{0, \pm 1\}$ and we can w.l.o.g. assume that $x_{k_0} = 1$.)

In a similar way we can get a lower estimate, which converges to 1 when $k_0 \to \infty$. $\square$

Exercise 13.F. Prove the following extension of the second part of (vii) of Theorem 13.1.

Let f be a differentiable function on $[0,1]$ such that $f'(x) = 0$ for every $x \in [0,1] \setminus \mathbb{D}$. Then f is constant. (Hint. For simplicity, assume that $f(0) = 0$, $f(1) = 1$. Find a sequence $[0,1] = [a_0, b_0] \supset [a_1, b_1] \supset \dots$ such that for each n, $b_n - a_n = 3^{-n}$ and $f(b_n) - f(a_n) \geq 2^{-n}$. Apply Exercise 7.O.)

It is interesting to observe that in Exercise 7.P(vi) we introduced 'the Cantor function' on $[0,1]$ which is not constant although its derivative vanishes everywhere on $[0,1] \setminus \mathbb{D}$. It follows that at some points of $\mathbb{D}$ the Cantor function cannot be differentiable.

Exercise 13.G. (Another extension of Theorem 13.1(vii)) Let $f\colon I \to \mathbb{R}$ be differentiable and suppose that $f'(x) = 0$ for all but countably many points x of f. Show that f is constant. (Hint. f' is Darboux continuous.)

Exercise 13.H. Le $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable and such that f is monotone on no interval. Let $N := \{x; f'(x) = 0\}$, $S := \{x; f' \text{ is continuous at } x\}$. Then $S \subset N$, N is dense, $\mathbb{R} \setminus N$ is dense and meagre.

Solution. $\boxed{S \subset N}$ Let $x_0 \in S$, i.e., f' is continuous at the point x_0. Suppose that $f'(x_0) \neq 0$. Then there is an $\varepsilon > 0$ such that $f'(x_0) \neq 0$ for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, which means that f is monotone on the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$. This is a contradiction.

Both $\mathbb{R} \setminus N$ and N are dense. Since f is not monotone in any interval, in every interval there exist point $x_{1,2}$, $y_{1,2}$ such that $x_i < y_i$, $f(x_1) < f(y_1)$ and $f(x_2) > f(y_2)$. By the mean value theorem, there exist ξ_1, ξ_2 between them such that $f'(\xi_1) > 0$ and $f'(\xi_2) < 0$. Since $\xi_{1,2} \in \mathbb{R} \setminus N$, this implies that $\mathbb{R} \setminus N$ is dense.

Darboux property of f' implies that there exists a point η between ξ_1 and ξ_2 such that $f'(\eta) = 0$. Therefore N is dense in $\mathbb{R}$, too.

$\mathbb{R} \setminus N$ is meager. We have $\mathbb{R} \setminus N \subseteq \mathbb{R} \setminus S$, so it suffices to show that $\mathbb{R} \setminus S$ is meager. Recall the function $g = f'$ is a function of first class (Exercise 11.C). By Theorem 11.4 points of discontinuity of a function of first Baire class form a meagre F_σ set; therefore $\mathbb{R} \setminus S$ is meager.
Another possibility: The set $\mathbb{R} \setminus N = g^{-1}(\mathbb{R} \setminus \{0\})$ is an F_σ-set (Theorem 11.6). To show that an F_σ-set is meager, it suffices to show that it has empty interior (Exercise 6.I). This is clear from the fact that its complement N is dense. $\square$

4.14 Derivatives

Exercise 14.A. Find an example of an $f \in \mathcal{DC} \cap \mathcal{B}^1$ without an antiderivative.

Exercise 14.B. By applying the foregoing to the functions j_1, j_2, j_3 with

$$\begin{aligned} j_1(x) &:= \sin 2\pi x \\ j_2(x) &:= |\sin 2\pi x| \\ j_3(x) &:= \sin^2 2\pi x \end{aligned}$$

show that the following statements are *false.*
(i) If $f \in \mathcal{D}'[0,1]$ then $|f| \in \mathcal{D}'[0,1]$.
(ii) If $f \in \mathcal{D}'[0,1]$ then $f^2 \in \mathcal{D}'[0,1]$.

Solution. Note that we get $A_1 = 0$ and $A_2, A_3 > 0$.

Example 14.1 then says that the functions

$$h_1 = \begin{cases} \sin 2\pi x^{-1} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

$$h_2 = \begin{cases} |\sin 2\pi x^{-1}| & \text{if } x > 0, \\ A_2 & \text{if } x = 0. \end{cases}$$

$$h_3 = \begin{cases} \sin^2 2\pi x^{-1} & \text{if } x > 0, \\ A_3 & \text{if } x = 0. \end{cases}$$

have antiderivatives.

If the function $|h_1|$ had antiderivative, then the function

$$h_2 - |h_1| = \begin{cases} 0 & \text{if } x > 0, \\ A_2 & \text{if } x = 0 \end{cases}$$

would have antiderivative, too. But this function cannot be a derivative, since it is not Darboux continuous.

Similar argument works for $h_3 - h_1^2$. □

Exercise 14.C. Define $j\colon [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} j(x) &:= \begin{cases} x & \text{if } 0 \le x < \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} < x < 1 \end{cases} \\ j(x) &= j(x+1) \text{ for all } x \ge 0 \end{aligned}$$

By applying the method of Example 14.1 to the functions $j+1$ and $1/(j+1)$ show that there exists an $f \in \mathcal{D}'[0,1]$ with $1 \le j \le 2$ and $1/j \notin \mathcal{D}'[0,1]$.

Solution. If we apply Example 14.1 to $j+1$ we get that the function

$$h_1(x) = \begin{cases} j(x^{-1}) + 1; & x \ne 0, \\ \frac{5}{4}; & x = 0, \end{cases}$$

is differentiable. (We have $\int_0^1 j(x)\,\mathrm{d}x = 2\int_0^{1/2}(x+1)\,\mathrm{d}x = \frac{5}{4}$.)

For the function $1/(j+1)$ we get that the function

$$h_2(x) = \begin{cases} \frac{1}{j(x^{-1})+1}; & x \neq 0, \\ 2\ln\frac{3}{2} & x = 0. \end{cases}$$

is differentiable. (We have $\int_0^1 \frac{1}{j(x^{-1})+1}\,\mathrm{d}x = 2\int_0^{1/2}\frac{1}{x+1}\,\mathrm{d}x = 2\ln\frac{3}{2}$.)

We see that h_2 coincide with $1/h_1$ for all points $x \neq 0$ but $h_2(0) \neq 1/h_1(0)$. This implies that $h_1(x)$ is not a differentiable function. □

Exercise 14.D. (We extend Exercises 14.B and 14.C.) Let $\phi\colon \mathbb{R} \to \mathbb{R}$. Prove the equivalence of the following two statements.
(α) If $f \in \mathcal{D}'[0,1]$, then $\phi \circ f \in \mathcal{D}'[0,1]$.
(β) There exist $\alpha, \beta \in \mathbb{R}$ such that ϕ is the function $x \mapsto \alpha x + \beta$ ($x \in \mathbb{R}$).
(Hint for the implication (α) $\Rightarrow$ (β). Prove that $\phi \in \mathcal{D}'(\mathbb{R})$. Let Φ be an antiderivative of ϕ. Let j be as in the previous exercise. Let $a, b \in \mathbb{R}$, $a \neq 0$. Apply the results of Example 14.1 to the functions $j_1 := 2aj + b$ and $j_2 := \phi \circ j_1$ and show that $a^{-1}\Phi(a+b) - a^{-1}\Phi(a) = \phi(\frac{1}{2}a + b)$ Thus,

$$\Phi(x+y) - \Phi(x-y) = 2y\phi(x) \qquad (x, y \in \mathbb{R}).$$

From this formula (and the relation $\Phi' = \phi$ property (β) can be deduced in many ways.)

Solution. (α) $\Rightarrow$ (β) Using the functions $f(x) = x + k/2$, $k \in \mathbb{Z}$, we get that ϕ is differentiable on $\mathbb{R}$.

For the function $j_1 = 2aj + b$ we have

$$\int_0^1 j_1(x)\,\mathrm{d}x = b + 2a\int_0^1 j(x)\,\mathrm{d}x = b + \frac{a}{2}.$$

We get that the function

$$u_{a,b}(x) = \begin{cases} b + 2aj(x^{-1}); & x \neq 0, \\ b + \frac{a}{2}; & x = 0. \end{cases}$$

is differentiable.

From the condition (α) we know that $j_2 = \phi \circ j_1 \in \mathcal{D}'$. For the function $j_2 = \phi \circ j_1$ we have (for $a \neq 0$)

$$\int_0^1 \phi(j_1(x))\,\mathrm{d}x = 2\int_0^{1/2}\phi(b + 2ax)\,\mathrm{d}x = \frac{1}{a}\int_b^{b+a}\phi(t)\,\mathrm{d}t = \frac{\Phi(a+b) - \Phi(a)}{a}.$$

Therefore the function

$$v_{a,b}(x) = \begin{cases} \phi(b + 2aj(x^{-1})); & x \neq 0, \\ \frac{\Phi(a+b)-\Phi(b)}{a} & x = 0. \end{cases}$$

is differentiable.

Again from the condition (α) we have that $\phi \circ u_{a,b} \in \mathcal{D}'$. It is also clear that $\phi \circ u_{a,b}(x) = v_{a,b}(x)$ for every $x \neq 0$.

We see, that if $\phi \circ u_{a,b}$ is in $\mathcal{D}'$, then necessarily

$$\phi\left(b + \frac{a}{2}\right) = \frac{\Phi(a+b) - \Phi(b)}{a}.$$

If we set $b = x - y$ and $a = 2y$, we get

$$\phi(x) = \frac{\Phi(x+y) - \Phi(x-y)}{2y} \tag{4.1}$$

or

$$2y\phi(x) = \Phi(x+y) - \Phi(x-y).$$

If we differentiate w.r.t y we get that

$$2\phi(x) = \phi(x+y) + \phi(x-y). \tag{4.2}$$

(Note that we have (4.1) only for $y \neq 0$, which corresponds to $a \neq 0$, but (4.2) is obviously true for $y = 0$, too.)

We can w.l.o.g assume that $\phi(x) = 0$. To see this, it suffices to show that if $\Phi' = \phi$ fulfills the equation (4.1), then so do the functions $\Phi(x) + cx$ and $\phi(x) + c$. Indeed we have

$$\frac{\Phi(x+y) + c(x+y) - \Phi(x-y) - c(x-y)}{2y} = \frac{\Phi(x+y) - \Phi(x-y)}{2y} + \frac{2cy}{2y} = \phi(x) + c.$$

So from now we assume that ϕ is a derivative, it fulfills (4.1) and $\phi(0) = 0$.

No that if we plug $x = 0$ or $y = x$ into (4.2), we get

$$\phi(-y) = -\phi(y);$$
$$\phi(2x) = 2\phi(x).$$

Using the above equations we get for $x = \frac{a+b}{2}$, $y = \frac{a-b}{2}$ that

$$\phi(a+b) = 2\phi\left(\frac{a+b}{2}\right) = \phi(a) + \phi(b).$$

So the function ϕ fulfills *Cauchy functional equation.*

Additionally, since ϕ is a derivative, it belongs to the first class of Baire and by Theorem 11.2, it is measurable. It is known that every measurable solution of Cauchy functional equation is of the form $\phi(x) = ax$.

A different argument:[24] From Theorem 11.4 we know that for every Baire function the set of discontinuity points is a meager F_σ-set. Hence f has at least one continuity point. It is known that if a solution of Cauchy functional equation is continuous at one point, it is continuous everywhere. □

[24] `http://math.stackexchange.com/questions/359183/measurable-cauchy-function-is-continuous`; shorter url: `http://math.stackexchange.com/q/359183`

Exercise 14.E. Prove the following extension of Theorem 14.2. Let I be an interval. Let $f_1, f_2, \dots$ be a sequence of elements of $\mathcal{D}'(I)$ that converges uniformly to a function $f\colon I \to \mathbb{R}$. Let $x_0 \in I$. For each n, let F_n be an antiderivative of f_n for which $F_n(x_0) = 0$. Then the sequence $F_1, F_2, \dots$ converges to an antiderivative of f. The convergence is uniform if I is bounded.

Solution. The first part is basically shown in the proof of Theorem 14.2.

To see that the convergence is uniform it suffices to verify the Cauchy-Bolzano condition. If the length of the interval I is $|I|$ we have for any x

$$\left|\sum_{n=k}^{k+p} G_n(x)\right| \le \sum_{n=k}^{k+p} |G_n(x)| \le \sum_{n=k}^{k+p} 2^{-n}|x - x_0| \le 2^{-k+1}|I|.$$

□

Exercise 14.F. (i) If $f \in \mathcal{D}'[a,b]$ is bounded and $g\colon [a,b] \to \mathbb{R}$ is continuous, then $fg \in \mathcal{D}'[a,b]$. (Hint. First show that $x \mapsto xf(x)$ is in $\mathcal{D}'$. Then use Weierstrass' approximation theorem (8.1(vii)) and Theorem 14.2.)
(ii) Let $f, g\colon [0,1] \to \mathbb{R}$ be defined as follows. $f(0) := g(0) := 0$. For $x > 0$, $f(x) := x^{-1/2} \sin x^{-1}$ and $g(x) := x^{1/2} \sin x^{-1}$. Then $f \in \mathcal{D}'$, g is continuous, but $fg \notin \mathcal{D}'$.

Solution of (i). $x \mapsto xf(x)$ is in $\mathcal{D}'$, whenever $f \in \mathcal{D}'$. Let $F' = f$. Notice that $(xF(x))' = xf(x) + F(x)$. Since F is an differentiable, it is continuous and thus it is in $\mathcal{D}'$ itself. Therefore $xf(x) = (xF)' - F$ belongs to $\mathcal{D}'$.

Using this fact we can inductively show that $f(x) \in \mathcal{D}'$ implies $P(x)f(x) \in \mathcal{D}'$ for any polynomial P.

Approximation of fg. By Weierstrass Theorem we know that there are polynomials P_n such that $\lim_{n\to\infty} P_n = g$ uniformly on $[a,b]$.

If we assume that f is bounded, i.e., $|f(x)| \le M$ for all x, then we get $|f(x)P_n(x) - f(x)g(x)| \le M|P_n(x) - g(x)|$; which means that fP_n converges uniformly to fg.

Since each fP_n belongs to $\mathcal{D}'$, we get from Theorem 14.2 that $fg \in \mathcal{D}'$. □

Solution of (ii). $f \in \mathcal{D}'$. Consider the function $F(x) = x^{3/2} \cos \frac{1}{x}$, $F(0) = 0$. We have

$$F'(x) = \left(x^{3/2} \cos \frac{1}{x}\right)' = \frac{3}{2}\sqrt{x} \cos \frac{1}{x} + \frac{\sin \frac{1}{x}}{\sqrt{x}}$$

for $x \ne 0$ and

$$F'(0) = \lim_{x\to 0} \frac{x^{3/2} \cos \frac{1}{x}}{x} = \lim_{x\to 0} \sqrt{x} \cos \frac{1}{x} = 0.$$

The function h defined by $h(x) = \frac{3}{2}\sqrt{x} \cos \frac{1}{x}$ for $x \ne 0$ and $h(0) = 0$ is continuous, thus $h \in \mathcal{D}'$. This implies that f, as a linear combination of two derivatives, belongs to $\mathcal{D}'$, too.

$fg \notin \mathcal{D}'$. We have

$$f(x)g(x) = \begin{cases} \sin^2 \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

If we apply Example 14.1 to the function $j(x) = \sin^2 x$, we see that a function having values $x \mapsto j(x^{-1})$ for $x \neq 0$ is a derivative if and only if its value at the point 0 is equal to

$$A = \frac{1}{4\pi^2} \int_0^{2\pi} j(x) \, \mathrm{d}x$$

and, clearly, $A > 0$. (For $j(x)$ which has period 1 we have $A = \int_0^1 j(x) \, \mathrm{d}x$. If $h(x)$ has period 2π, we can take $j(x) = h(2\pi x)$. Then $\int_0^1 j(x) \, \mathrm{d}x = \frac{1}{2\pi} \int_0^{2\pi} h(x) \, \mathrm{d}x$. And $H'(x) = J'(x/2\pi) = \frac{1}{2\pi} j(x/2\pi) = \frac{1}{2\pi} h(x)$.) □

Exercise 14.G. Let $f \in \mathcal{D}'[a,b]$ be bounded and let $g\colon [0,1] \to [a,b]$ have a continuous derivative with $g'(x) > 0$ for all $x \in [0,1]$. Then $f \circ g \in \mathcal{D}'[0,1]$ (apply Exercise 14.F).

Solution. Let $F' = f$. We have

$$(F \circ g(x))' = f(g(x))g'(x)$$

and thus

$$f \circ g = (F \circ g)' \cdot \frac{1}{g'}$$

is a product of a bounded[25] function from $\mathcal{D}'$ and a continuous function. So it belongs to $\mathcal{D}'$ by Exercise 14.F. □

Exercise 14.H. (An extension of Exercise 14.F(i))
(i) Let $h \in \mathcal{D}'[0,1]$ be bounded and such that $h \geq 0$, $h(0) = 0$ while h is continuous at every point of $(0,1]$. Then $fh \in \mathcal{D}'[0,1]$ for every bounded $f \in \mathcal{D}'[0,1]$. (Hint. Let $f \in \mathcal{D}'$, $0 \leq f \leq 1$. Use Exercise 14.F to construct a function G on $(0,1]$ for which $G' = fh$. Show that

$$0 \leq G(y) - G(x) \leq H(y) - H(x) \qquad (0 \leq x \leq y \leq 1)$$

where $H\colon [0,1] \to \mathbb{R}$ is an antiderivative of h. Deduce that G can be extended to a continuous function G_1 on $[0,1]$ for which $G_1'(0) = 0 = f(0)h(0)$.)
(ii) Find a function h having the properties mentioned in (i) without being continuous at 0. (For example, first find a continuous function h on $(0,1]$ with $h(1/n) = 1$ for every $n \in \mathbb{N}$ and $\int_0^x h(t) \, \mathrm{d}t \leq x^2$ for all $x \in (0,1]$.)

[25] The function $(F \circ g)' = f \circ g \cdot g'$ is bounded, since $f \circ g$ is bounded and g' is continuous function on a compact interval, so it is bounded, too.

Solution for (i). From continuity of h we get, using Exercise 14.F, that there exists a function $G\colon [\varepsilon, 1] \to \mathbb{R}$ such that $G'(x) = f(x)h(x)$ for each $x \geq \varepsilon$. Such function is clearly unique up to a constant, so by choosing a common value for $G(1)$ we get the same function on the interval $[\varepsilon, 1]$ and by putting these functions together we have $G\colon (0, 1] \to \mathbb{R}$ such that $G'(x) = f(x)h(x)$ for each $x > 0$.

Now, if we assume that $0 \leq f \leq 1$, then we have $0 \leq G' \leq H'$. Thus for the function $J = H - G$ we have $J' \geq 0$ and thus Mean Value Theorem implies $J(y) \geq J(x)$ for $0 < x \leq y \leq 1$. This is the same as

$$0 \leq G(y) - G(x) \leq H(y) - H(x) \tag{4.3}$$

for any $0 < x \leq y \leq 1$.

Using the above fact we get that

$$0 \leq \lim_{\delta \to 0^+} \sup\{|G(y) - G(x)|; 0 < x, y < \delta\} \leq \lim_{\delta \to 0^+} \sup\{|H(y) - H(x)|; 0 < x, y < \delta\} = 0,$$

since $\lim_{x \to 0} H(x) = H(0)$ (the function H is differentiable and thus continuous).

This shows that G can be continuously extended[26] to a continuous function $G_1\colon [0, 1] \to \mathbb{R}$ by putting $G_1(0) = \lim_{x \to 0^+} G(x)$.

Now taking the limit $x \to 0^+$ in (4.3) gives

$$\begin{aligned} 0 \leq G_1(y) - G_1(0) &\leq H(y) - H(0) \\ 0 \leq \frac{G_1(y) - G_1(0)}{y} &\leq \frac{H(y) - H(0)}{y} \end{aligned}$$

and since the limit of the RHS for $y \to 0^+$ is equal to $H'(0) = h(0) = 0$, we get

$$G_1'(0) = \lim_{y \to 0^+} \frac{G_1(y) - G_1(0)}{y} = 0 = f(0)h(0).$$

□

Solution for (ii). For example, we can define function separately on each interval $[1/(n+1), 1/n]$. We choose the function on the interval $[a, b]$ as a piecewise linear function

$$h(x) = \begin{cases} 0 & a \leq x \leq b - \Delta, \\ \frac{2}{\Delta} x - \frac{2}{\Delta}(b - \Delta) & b - \Delta \leq x \leq b - \Delta/2, \\ b\frac{2}{\Delta} - \frac{\Delta}{x} & b - \Delta/2 \leq x \leq b; \end{cases}$$

i.e. our function is non-zero only on the interval $[b - \Delta, b]$, and on this interval, the graph is an isosceles triangle of height 1.

[26] We are using the fact that a function has a limit at a point x_0 if and only if the oscillation of f at this point is equal to zero. This is sometimes called Cauchy criterion for functions; see, for example, [Z, p.132, Theorem 4].

The only remaining question is how to choose Δ's in a such way that $\int_0^x h(t)\ \mathrm{d}t \le x^2$. To this end, it suffices if we have $\int_a^t h(t)\ \mathrm{d}t \le t^2 - a^2$ for $t \in [a, b]$. This is certainly true if $2\Delta \le (b-\Delta)^2 - a^2$, which yields the inequality $\Delta^2 + 2(1+b)\Delta \le b^2 - a^2$.

The function h defined in this way is non-negative, continuous on $(0, 1]$, but it is not continuous at 0. Since there is only one point of discontinuity, the function is Riemann integrable. If we define $H(x) = \int_0^x h(t)\ \mathrm{d}t$, then $H'(x) = h(x)$ for $x \in [0, 1]$ (since the function h is continuous on this interval, see Theorem 8.1(viii)). And we also have

$$0 \le H'(0) = \lim_{x\to 0} \frac{\int_0^x h(t)\ \mathrm{d}t}{x} \le \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0,$$

i.e. $H'(0) = 0$.

So if we define $h(0) = 0$, we obtain $h \in \mathcal{D}'$. □

Exercise 14.I. If f has a continuous derivative and if $g \in \mathcal{D}'$ then $fg \in \mathcal{D}'$. (Observe that here we have no boundedness condition. Compare Exercises 14.F and 14.H.)

Solution. Let $G' = g$. Then we have $(fG)' = fg + f'G$.

Since both f' and G are continuous, the function $f'G$ is continuous and thus $f'G \in \mathcal{D}'$.

Then fg is a difference of two functions from $\mathcal{D}'$, so it belongs to $\mathcal{D}'$, too. □

Exercise 14.J. A curious property of derivatives is that they all have connected graphs. More generally, if $f\colon \mathbb{R} \to \mathbb{R}$ is Darboux continuous and of the first class of Baire, then its graph Γ is connected.

We outline a proof. Suppose there exist open subsets U_1, U_2 of $\mathbb{R}^2$ such that $U_1 \cup U_2 \supset \Gamma$, $U_1 \cap U_2 = \emptyset$, $U_1 \cap \Gamma \ne \emptyset$, $U_2 \cap \Gamma \ne \emptyset$. For $i = 1, 2$ let $A_i = \{x \in \mathbb{R}; (x, f(x)) \in U_i\}$. Then $A_1 \cup A_2 = \mathbb{R}$, $A_1 \cap A_2 = \emptyset$. Put $X := \overline{A_1} \cap \overline{A_2}$.

(i) If $p \in A_1$, $q \in A_2$ and $p < q$, then $[p, q]$ intersects X.

(ii) If $a \in A_1$, then there exists $a_1, a_2, \dots \in A_1$ such that $a_1 < a_2 < \dots$ and $\lim\limits_{n\to\infty} a_n = a$.

(iii) If J is a component of $\mathbb{R} \setminus X$, then $\overline{J} \subset A_1$ or $\overline{J} \subset A_2$. (This follows from obvious variations of (i) and (ii).)

(iv) Let g be the restriction of f to X. There exists a $c \in X$ such that g is continuous at c. Without loss of generality, assume $c \in A_1$. Then there is an $\varepsilon > 0$ with $X \cap (c-\varepsilon, c+\varepsilon) \subset A_1$.

(v) All components of $(c-\varepsilon, c+\varepsilon) \setminus X$ are contained in A_1.

(vi) $c \in \overline{A}_2$; a contradition.

Proof. (i) Let $x_0 = \inf\{x \in [p, q]; x \in A_2\}$. Then $x_0 \in \overline{A_2}$ by definition. Simultaneously, we have either $x_0 = p$ or $[p, x_0) \subseteq A_1$, in either case $x_0 \in \overline{A_1}$.

The same argument works for $x_1 = \sup\{x \in [p, q]; x \in A_1\}$.

(ii) We have $(a, f(a)) \in U_1$. Since U_1 is open in $\mathbb{R}^2$, there exists $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \times (f(a)-\varepsilon, f(a)+\varepsilon) \subseteq U_1$. If there was no point from A_1 in the interval $(a-\varepsilon, a)$, then we would have $f(x) \notin (f(a)-\varepsilon, f(a)+\varepsilon)$ for each $x \in (a-\varepsilon, \varepsilon)$. This contradicts Darboux continuity of f. (If f is Darboux continuous, then $f[(a-\varepsilon, a]]$ is connected.)

Repeating this argument for some sequence $\varepsilon_n \searrow 0$ we get the desired sequence.

(iii) Connected subsets of $\mathbb{R}$ are singletons and intervals. If J is a singleton, then $\overline{J} = J$.

If J is an interval, then it must lie entirely in A_1 or entirely in A_2. (If there were points from both sets A_1 and A_2 in the interval J, then we would get from (i) that $J \cap X \neq \emptyset$.)

It remains to notice that if $(a, b) \subseteq A_1 \Rightarrow$, then we get $[a, b] \subseteq A_1$ using (ii). (For example, if $a \in A_2$, then we would get a contradiction because (ii) implies existence of a sequence of points from A_2 such that $a_n \searrow a$.)

(iv) The existence of the point c follows from Baire characterization theorem: A real function belongs to first Baire class if and only if, for every non-empty closed subset C of $\mathbb{R}$, the restriction of f to $\mathbb{R}$ has a continuity point. (Appendix C) Although here we need only the easier implication, which can be shown basically in the same way as Theorem 11.4.

Since $c \in A_1$, we have $(c, f(c)) \subseteq U_1$. This implies that there is an $\delta_1 > 0$ such that $(c-\delta_1, c+\delta_1) \times (f(c)-\delta_1, f(c)+\delta_1) \subseteq U_1$.

The continuity of $f|_X$ in c implies that there exists δ_2 such that $f(x) \in (f(c)-\delta_1, f(c)+\delta_1)$ whenever $|x-c| < \delta_2$ and $x \in X$.

If we take $\varepsilon = \min\{\delta_1, \delta_2\}$, then we get $(c, f(c)) \in U_1$ for each $x \in (c-\varepsilon, c+\varepsilon) \cap X$. This means that $(c-\varepsilon, c+\varepsilon) \cap X \subseteq A_1$.

(v) A connected subset of $\mathbb{R}$ can be either a point or an interval.

Again, we have either $C \subseteq A_1$ or $C \subseteq A_2$ (the argument is the same as in (iii)).

Suppose that there is a component C of $(x-\varepsilon, x+\varepsilon) \setminus X$ such that $C \subseteq A_2$.

If the component is of the form $C = \{x\}$, $x \in A_2$, then necessarily $x \in \overline{A_1}$ (otherwise there would be a larger interval contained in A_2). This yields $x \in X \cap A_2$ a contradiction.

If the component is interval, then by (iii) this interval is closed, i.e. it has the form $[a, b]$. Again we get $a, b \in \overline{A_1}$ (otherwise the component could be enlarged.) Then we have $a, b \in X$, which is again a contradiction.

(vi) We have both $(c-\varepsilon, c+\varepsilon) \subseteq A_1$ and $X \setminus (c-\varepsilon, c+\varepsilon) \subseteq A_1$. This gives together $(c-\varepsilon, c+\varepsilon) \subseteq A_1$, which contradicts $c \in \overline{A}_2$. □

A different proof is given in [Br, Theorem II.1.1], [BC, Theorem 4.1].

Proof. Let $U_{1,2}$, $A_{1,2}$ and $X = \overline{A_1} \cap \overline{A_2} = \delta A_1 = \delta A_2$ has the same meaning as above.

Clearly, *X is a closed set.*

The set X has no isolated points.

Suppose that, on the contrary, there is some $x \in X$ and $\varepsilon > 0$ such that $X \cap (x-\varepsilon, x+\varepsilon) = \emptyset$. W.l.o.g. let $x \in A_1$. Since, $x \in \overline{A_2}$, there is a sequence $b_n \in A_2$ such that $b_n \to x$. W.l.o.g. we can assume $b_n \searrow x$.

At the same time, we have from (i) that there is a sequence $a_n \searrow x$ such that $a_n \in A_1$.

Using (ii) we see that between each a_n and b_n there always is a point $b_n \in X$. This implies that $(x-\varepsilon, x+\varepsilon) \setminus \{x\}$ contains a point from X.

Both $X \cap A_1$ and $X \cap A_2$ are dense in X. Suppose that there is an $x \in X$ such that $x \notin \overline{X \cap A_2}$. Notice that we necessarily have $x \in A_1$ and $x \in \overline{A_2}$.

The assumption $x \notin \overline{X \cap A_2}$ implies

$$(x-\varepsilon, x+\varepsilon) \cap X \subseteq A_1 \tag{A1}$$

for some $\varepsilon > 0$.

Now since $x \in \overline{A_2}$, there must be sume $y \in A_2$ which belongs to the open interval $(x-\varepsilon, x+\varepsilon)$. W.l.o.g. let $y > x$.

We have $y \notin X$ from (A1), and thus $y \notin \overline{A_1}$. This means that there is an open interval containing y which lies entirely in A_2 and misses A_1.

Define

$$z := \inf\{t; (t, y] \subseteq A_2\}.$$

We have $x < z < y$.

The point z must belong either to A_1 or to A_2.

If $z \in A_1$, then by (ii) we have a sequence $z_n \searrow z$, $z_n \in A_1$. Some of the terms of this sequence must belong to $[z, y]$, which contradicts the assumption $(z, y] \subseteq A_2$.

Now assume that $z \in A_2$. We must have $z \in \overline{A}_1$ (otherwise there is a neighborhood of z missing the set A_1 and z is not the infimum – it can be lowered). Thus $z \in A_2 \cap \overline{A_1}$, i.e., $z \in X$, which contradicts (A1).

The restriction $f|_X$ has a point of continuity $x_0 \in X$. This follows from Baire characterization theorem: A real function belongs to first Baire class if and only if, for every non-empty closed subset C of $\mathbb{R}$, the restriction of f to $\mathbb{R}$ has a continuity point. (Appendix C) Although here we need only the easier implication, which can be shown basically in the same way as Theorem 11.4.

Continuity of $f|_X$ at $x_0 \in X$ contradicts density of both $X \cap A_1$ and $X \cap A_2$ in X. Suppose that $x_0 \in A_1$, i.e. $f(x_0) \in U_1$. Again, we can choose some open subsets of $\mathbb{R}$ such that $(x_0, f(x_0)) \in A_1 \times B_1 \subseteq U_1$. The continuity of f at x_0 implies that there is a neighborhood $U = X \cap (x_0-\varepsilon, x_0+\varepsilon)$ such that $f(x) \in B_1$ for each $x \in U$. This implies that $U \subseteq A_1$ and thus this neighborhood of x_0 in X contains no points from A_2. This contradicts the density of the set $X \cap A_2$ in X. $\square$

Perhaps it is worth recalling that Darboux functions with disconnected graph do exist, see Exercise 9.C.

4.15 The fundamental theorem of calculus

Exercise 15.A. For $f, g \in \mathcal{R}[a,b]$ the following conditions are equivalent.
(α) $Jf = Jg$, i.e., $\int_a^x f(t)\,\mathrm{d}t = \int_a^x g(t)\,\mathrm{d}t$.
(β) If $c, d \in [a,b]$, then $\int_c^d f(t)\,\mathrm{d}t = \int_c^d g(t)\,\mathrm{d}t$.
(γ) $f(x) = g(x)$ for every point x of $[a,b]$ where both f and g are continuous.
(δ) $f = g$ a.e. on $[a,b]$.
(ε) $\{x\colon f(x) = g(x)\}$ is dense in $[a,b]$.
(ζ) $\int_a^b |f(x) - g(x)| = 0$.
(Prove (α) $\Leftrightarrow$ (β) $\Rightarrow$ (γ) $\Rightarrow$ (δ) $\Rightarrow$ (ε) $\Rightarrow$ (ζ) $\Rightarrow$ (β) using the results of Section 12.)

Proof of (ζ) $\Rightarrow$ (β). If $|f(x) - g(x)| = 0$ on a dense set, then $\int_c^d |f(t) - g(t)|\,\mathrm{d}t = 0$. (See Exercise 12.F.)

Since

$$0 \le \left| \int_c^d |f(t)| - |g(t)| \right| \,\mathrm{d}t \le \int_c^d |f(t) - g(t)|\,\mathrm{d}t = 0$$

we get that $\int_c^d |f(t)|\,\mathrm{d}t = \int_c^d |g(t)|\,\mathrm{d}t$.

Now since both f and g are bounded and the condition (ζ) is not influenced by replacing f, g by $f + \overline{C}$ and $g + \overline{C}$ for a real constant C, and the same is true for the condition (β), it suffices to prove this for non-negative functions f, g. □

5 Borel measurability

5.16 The classes of Baire

The following observation might be useful:[27]

Lemma. *If functions f_n belong to $\mathcal{B}^\alpha$ and $f(x) = \limsup_{n\to\infty} f_n(x)$, then $f \in \mathcal{B}^{\alpha+2}$.*

Proof. We have

$$f(x) = \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\sup_{k\ge n} f_k(x) \right)$$

and

$$\sup_{k\ge n} f_k(x) = \lim_{m\to\infty} \max_{k\le n\le m} f_k(x).$$

It suffices to use the fact that $\mathcal{B}^\alpha$ is closed under *finite* maxima. □

[27] See also [N, Theorem XV.2.3, p.135]

Exercise 16.A. Show that the $\phi\colon \mathbb{R} \to \mathbb{R}$ constructed in Exercise 9.M is in $\mathcal{B}^3$.

The function ϕ in 9.M is defined in this way: For $x \in [0,1]$ let $0.x_1x_2x_3\ldots$ be the standard dyadic development of $x - \lfloor x \rfloor$:

$$x_n = \lfloor 2^n x \rfloor - 2\lfloor 2^{n-1} x \rfloor$$

where $\lfloor x \rfloor$ is the entire part of x. We define $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Solution. Since $x \mapsto \lfloor x \rfloor$ is in $\mathcal{B}^1$, we get that $f_n\colon x \mapsto \frac{x_1+x_2+\cdots+x_n}{n}$ belongs to $\mathcal{B}^1$ as well. (Each class $\mathcal{B}^\alpha$ is closed under addition and multiplication.)

Since $\phi = \limsup_{n\to\infty} f_n$, we get $\phi \in \mathcal{B}^3$ from the above lemma. □

Exercise 16.B. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that

$$D_r f(x) := \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}$$

exists for every $x \in \mathbb{R}$. Show that $D_r f \in \mathcal{B}^2$. (Hint: Use Theorem 7.7.)

Solution. If $\lim\limits_{y\to x^+} \frac{f(y)-f(x)}{y-x}$ exists for each x, then also $f(x+) = \lim\limits_{y\to x^+} f(y)$ exists for each x, since

$$f(y) = f(x) + \frac{f(y) - f(x)}{y - x} \cdot (y - x).$$

(In fact, it is equal to $f(x)$.)

From Theorem 7.7 we know that there exist only countably many points x of $\mathbb{R}$ for which f is not continuous at x but $f(x+)$ exists. (For arbitrary function f.)

According to Theorem 11.8, if a function has only countably many points of discontinuity, then it belongs to $\mathcal{B}^1$. □

Exercise 16.C. Let $f_1, f_2, \ldots$ be a sequence of continuous functions such that $f := \limsup f_n(x)$ is finite everywhere. Then $f \in \mathcal{B}^2$.

Solution. This follows from the above lemma.

On the other hand, the above lemma can be obtained from this exercise and Exercise 16.D. □

Exercise 16.D. (i) $f \in \mathcal{C}$, $g \in \mathcal{B}^m$ implies $f \circ g \in \mathcal{B}^m$
(ii) $f \in \mathcal{B}^n$, $g \in \mathcal{B}^m$ implies $f \circ g \in \mathcal{B}^{n+m}$

Solution. (i) By induction on m. *Inductive step.* If $g = \lim_{n\to\infty} g_n$, then $f \circ g = f \circ (\lim_{n\to\infty} g_n) = \lim_{n\to\infty}(f \circ g_n)$, if f is continuous.

$$f(g(x)) = f(\lim_{n\to\infty} g_n(x)) = \lim_{n\to\infty} f(g_n(x)).$$

If each g_n belongs to $\mathcal{B}^m$, then also each $f \circ g_n$ belongs to $\mathcal{B}^m$ by inductive hypothesis. This shows that $f \circ g \in \mathcal{B}^{m+1}$.

(ii) By induction on n.
1° If $n = 0$, then $f \in \mathcal{C}$ and this was shown in part (i).
2° Suppose that the claim holds for n and that $f \in \mathcal{B}^{n+1}$. Then $f = \lim_{k\to\infty} f_k$ for some $f_k \in \mathcal{B}^n$. We get

$$f(g(x)) = \lim_{k\to\infty} f_k(g(x)),$$

i.e. $f \circ g = \lim_{k\to\infty} f_k \circ g$. By inductive hypothesis we have $f_k \circ g \in \mathcal{B}^{n+m}$ and thus $f \circ g \in \mathcal{B}^{n+m+1}$. □

Exercise 16.E. (i) Let $X \subseteq \mathbb{R}$ be an F_σ or G_δ. Then $\xi_X \in \mathcal{B}^2$.
(ii) If $\xi_X \in \mathcal{B}^2$ then X is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$.
(iii) If X is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$, then $\xi_X \in \mathcal{B}^2$.
(Hint: Use Lemma 11.11.)

Solution. (i) Let $X = \bigcup X_n$, where X_n is closed. W.l.o.g. we can assume $X_n \subseteq X_{n+1}$. Every closed subset of $\mathbb{R}$ is G_δ. Hence $\xi_{X_n} \in \mathcal{B}^1$ according to Theorem 11.6. (Since each X_n is both F_σ and G_δ.) Therefore

$$f_X = \lim_{n\to\infty} f_{X_n}$$

belongs to $\mathcal{B}^2$.

(ii) We know that $g \in \mathcal{B}_1$ if and only if $g^{-1}(U)$ is F_σ for each open set U (or, equivalently, $g^{-1}(F)$ is G_δ for each closed set F); see Theorem 11.12.

Now if $\xi_X = \lim_{n\to\infty} f_n(x)$, where $f_n \in \mathcal{B}_1$, then the set

$$X = \xi_X^{-1}((1/2,\infty)) = \xi_X^{-1}([1/2,\infty)) = \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} f_n^{-1}([1/2,\infty))$$

is an $G_{\delta\sigma}$-set.

Using the same argument with the set $(-\infty, 1/2)$ we get that the complement $\mathbb{R} \setminus X$ is a $G_{\delta\sigma}$ set, which means that X is an $F_{\sigma\delta}$ set.

(iii) First let us notice that if we have $A \subseteq B$, where A is G_δ and B is F_σ, then by there is a set C such that $A \subseteq C \subseteq B$ and C is both F_σ and G_δ.

In the situation as above we have two F_σ set B and $\mathbb{R} \setminus A$ such that $B \cup (\mathbb{R} \setminus A) = \mathbb{R}$. If we apply Lemma 11.11 to these two F_σ sets we get that there exists F_σ set $C \subseteq B$ and $\mathbb{R} \setminus C \subseteq \mathbb{R} \setminus A$. This clearly implies $A \subseteq C$. Since $\mathbb{R} \setminus C$ is F_σ, the set C is also G_δ.

Let X be any set which is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$. Then we have $X = \bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n$ where each A_n is G_δ and each B_n is F_σ. We can, moreover, assume that $A_1 \subseteq A_2 \subseteq \cdots \subseteq X$ and $B_1 \supseteq B_2 \supseteq \cdots \supseteq X$.

Now for each $n \in \mathbb{N}$ we have C_n such that C_n is both F_σ and G_δ and also $A_n \subseteq C_n \subseteq B_n$. This implies

$$\xi_{A_n} \leq \xi_{C_n} \leq \xi_{B_n}.$$

Since we have

$$\xi_X = \lim_{n\to\infty} \xi_{A_n} = \lim_{n\to\infty} \xi_{B_n}$$

we also get

$$\xi_X = \lim_{n\to\infty} \xi_{C_n}.$$

From Theorem 11.6 we know that each ξ_{C_n} is in $\mathcal{B}^1$. So this shows that ξ_X belongs to $\mathcal{B}^2$. □

Exercise 16.F. Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Suppose that for every $y \in \mathbb{R}$ the set $\{x; f(x) = y\}$ is finite. For $y \in \mathbb{R}$, let $N_f(y)$ be the number of elements of $\{x; f(x) = y\}$. Thus we obtain a function N_f on $\mathbb{R}$ with values in $\mathbb{N} \cup \{0\}$ (see Exercise 9.N). Let Y_0 be the set of all local extrema of f. Recall that Y_0 is countable (Theorem 7.2).
(i) Show that $f(a) \in Y_0$ and $f(b) \in Y_0$.
(ii) Let $c \in \mathbb{R}$. Prove that there exists an $\varepsilon > 0$ such that $N_f(y) \geq N_f(c)$ for all[28] $y \in (c-\varepsilon, c+\varepsilon) \setminus Y_0$. (Hint. Let $x_1, \ldots, x_n$ be the elements of $\{x; f(x) = y\}$, $x_1 < x_2 < \cdots < x_n$. Choose $a_0, \ldots, a_N \in [a,b]$ in such a way that $a_0 < x_1 < a_1 < x_2 < \cdots < a_{n_1} < x_n < a_n$. Take $\varepsilon := \min_i |f(a_i) - c|$.)
(iii) Prove that the function N_f is of second class. (Define

$$g(y) := \begin{cases} \arctan N_f(y) & \text{if } y \in \mathbb{R} \setminus Y_0, \\ \frac{1}{2}\pi & \text{if } y \in Y_0. \end{cases}$$

Then $g(y) = g^{\uparrow}(y)$ for every $y \in \mathbb{R} \setminus Y_0$. Infer that the function $y \mapsto \arctan N_f(y)$ is of the second class and, consequently, so is N_f.)

Solution. TODO □

Even if the function N_f attains finite values, it can be unbounded; see Figure 4. (So we need to use rescaling with arctan or some similar function; it is not sufficient simply using some upper bound for the function N_f.) The same example shows that it can happen that function f is not constant in a neighborhood of c. (So we cannot proof $N_f(y) = N_f(c)$ instead of $N_f(y) \geq N_f(c)$.)

[28] I think that author meant here that $c \notin Y_0$ and $y \in (c-\varepsilon, c+\varepsilon)$. See Figure 3 for a counterexample showing that this claim is not true for $c \in Y_0$.

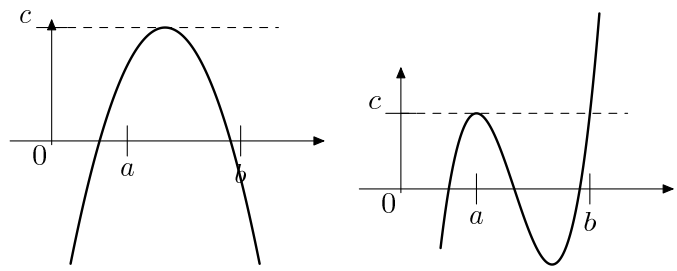

Figure 3: Figure illustrating counterexample to the claim in Exercise 16.F for $c \in Y_0$

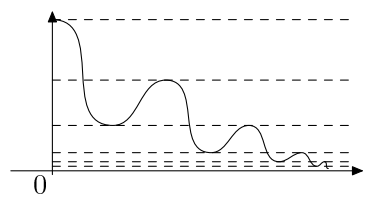

Figure 4: Sketch of a function illustrating Exercise 16.F

$\mathcal{B}$ is closed under addition, multiplication, maxima, minima. This is always done in two steps.

- Let $f \in \mathbb{C}$. Let $M = \{g\colon \mathbb{R} \to \mathbb{R}; f \vee g \in \mathcal{B}\}$. Clearly $\mathcal{C} \subseteq M$ and if $g_n \in M$, $g_n \to g$, then also $f \vee g = \lim\limits_{n\to\infty} (f \vee g_n)$ belongs to M. This shows that M is an L-set and thus $\mathcal{B} \subseteq M$. So we have shown that $f \in \mathcal{C}$, $g \in \mathcal{B}$ $\Rightarrow f \vee g \in \mathcal{B}$.
- Now fix $g \in \mathcal{B}$ and $N = \{f\colon \mathbb{R} \to \mathbb{R}; f \vee g \in \mathcal{B}\}$. In the first part we have shown that $\mathcal{C} \subseteq N$. If $f_n \in N$ and $f_n \to f$, then also $f \vee g = \lim\limits_{n\to\infty} (f_n \vee g)$ belongs to N. Thus N is an L-set and therefore, $\mathcal{B} \subseteq N$.

The second part shows that $f \vee g \in \mathcal{B}$ whenever $f, g \in \mathcal{B}$.

Note that we could show by transfinite induction that each $\mathcal{B}^\alpha$ is closed under $\vee$ (or the other operations mentioned here). This would give another proof of this fact.

$\mathcal{B}$ is closed under composition. We will try a similar approach as in the preceding part.

- Let $f \in \mathcal{C}$. Let $M = \{g\colon \mathbb{R} \to \mathbb{R}; f \circ g \in \mathcal{B}\}$. Clearly $\mathcal{C} \subseteq M$. And if $g_n \in M$, $g_n \to g$ then the continuity of f implies

$$f(g(x)) = f(\lim_{n\to\infty} g_n(x)) = \lim_{n\to\infty} f(g_n(x));$$
$$f \circ g = f \circ (\lim_{n\to\infty} g_n) = \lim_{n\to\infty} (f \circ g_n).$$

Therefore $g \in M$. Thus M is an L-set and $\mathcal{B} \subseteq M$. So we have shown that $f \in \mathcal{C}$, $g \in \mathcal{B}$ implies $f \circ g \in \mathcal{B}$.

- Now fix a function $g \in \mathcal{B}$ and define $N = \{f\colon \mathbb{R} \to \mathbb{R}; f \circ g \in \mathcal{B}\}$. The first part implies that $\mathcal{C} \subseteq N$. If $f_n \in N$ and $f_n \to f$ then $f(g(x)) = \lim\limits_{n\to\infty} f_n(g(x))$, i.e.,

$$f \circ g = (\lim_{n\to\infty} f_n) \circ g = \lim_{n\to\infty} (f_n \circ g).$$

So we get that $f \circ g \in \mathcal{B}$ and thus $f \in N$. We have shown that N is an L-set, which implies $\mathcal{B} \subseteq N$.

In the second part we have in fact shown $f, g \in \mathcal{B} \Rightarrow f \circ g \in \mathcal{B}$.

Exercise 16.G. Let $\mathcal{A}$ be a collection of subsets of $\mathbb{R}$ with the following two properties,
(i) If $A_1, A_2, \dots \in \mathcal{A}$, then $\bigcup_n A_n \in \mathcal{A}$ and $\bigcap_n A_n \in \mathcal{A}$.
(ii) Every closed subset of $\mathbb{R}$ is an element of $\mathcal{A}$.
Then every Borel set is an element of $\mathcal{A}$. (To prove this show that the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ with

$$\{x; f(x) \geq a\} \in \mathcal{A} \text{ for all } a \in \mathbb{R}$$

is an L-set.)

We see that we could also have defined Ω as the smallest $\mathcal{A}$ having the properties (i) and (ii).

Solution. Let $\mathcal{F} = \{f\colon \mathbb{R} \to \mathbb{R}; f^{-1}([a,\infty)) \in \mathcal{A} \text{ for all } a \in \mathbb{R}\}$. We want to prove that $\mathcal{F}$ is an L-set.

We have $\mathcal{C} \subseteq \mathcal{F}$, since $f^{-1}([a,\infty))$ is closed for a continuous map f.

Now let $f(x) = \lim\limits_{n\to\infty} f_n(x)$ where each $f_n \in \mathcal{A}$. We have

$$\begin{aligned} f(x) \geq a &\Leftrightarrow (\forall \varepsilon > 0) f(x) > a - \varepsilon \\ &\Leftrightarrow (\forall \varepsilon > 0) f(x) \geq a - \varepsilon \end{aligned}$$

Hence $f(x) \geq a$ can be equivalently expressed as

$$\begin{aligned} &(\forall \varepsilon > 0)(\exists m \in \mathbb{N})(\forall n > m) f_n(x) > a - \varepsilon \\ &(\forall \varepsilon > 0)(\exists m \in \mathbb{N})(\forall n > m) f_n(x) \geq a - \varepsilon \end{aligned}$$

So we have

$$f^{-1}([a,\infty)) = \bigcap_{\substack{\varepsilon > 0 \\ \varepsilon \in \mathbb{Q}}} \bigcup_{m \in \mathbb{N}} \bigcap_{n > m} f_n^{-1}([a - \varepsilon, \infty)).$$

This set is in $\mathcal{A}$ and this shows that $\mathcal{F}$ is an L-set.

This implies that $\mathcal{F} = \mathcal{B}$ and $\mathcal{A} = \Omega$. $\square$

Not all sets/functions are Borel measurable The author writes: "It takes a very sophisticated argument to show that not all functions and sets are Borel measurable. See 18.20 and Example 20.10." I think that this can be shown using cardinality.[29]

The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ has cardinality $2^{\mathfrak{c}}$.

On the other hand $\mathcal{B}_0 = \mathcal{C}$ has cardinality $\mathfrak{c}$.

If $\operatorname{card} \mathcal{A} = \mathfrak{c}$ that the set $\mathcal{A}^*$ of all pointwise limits of functions from $\mathcal{B}$ has cardinality at most $\operatorname{card}(\mathcal{A}^{\mathbb{N}}) = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Therefore (by transfinite induction) each $\mathcal{B}^\alpha$ has cardinality $\mathfrak{c}$.

And we have $\operatorname{card} \mathcal{B} = \operatorname{card}(\bigcup_{\alpha<\omega_1} \mathcal{B}^\alpha) \le \aleph_1 \cdot \mathfrak{c} = \mathfrak{c}$.

Exercise 16.H. Consider all sets $\mathcal{F}$ of functions $\mathbb{R} \to \mathbb{R}$ for which
(i) $\mathcal{C} \subset \mathcal{F}$;
(ii) if $f_1, f_2, \dots \in \mathcal{F}$ and the series $\sum f_n$ converges, then $\sum_{n=1}^\infty f_n \in \mathcal{F}$.
Show that among these sets there is a smallest one and that this smallest set is, in fact, $\mathcal{B}$. (Hint. First, show that $\mathcal{B}$ is such an $\mathcal{F}$. Each $\mathcal{F}$ is closed under addition and the intersection, $\mathcal{B}_1$, of all sets $\mathcal{F}$ is a group under addition. Now show that $\mathcal{B}_1$ is closed under pointwise limits. Use the fact that, if $f = \lim\limits_{n\to\infty} f_n$, then $f = f_1 + \sum_{n=1}^\infty (f_{n+1} - f_n)$.

Solution. Let $\mathcal{B}_1$ be then intersection of all sets $\mathcal{F}$ fulfilling the properties (i) and (ii). The set $\mathcal{B}_1$ fulfills (i), (ii), too.

We also have $f_1, f_2 \in \mathcal{B}_1 \Rightarrow f_1 + f_2 \in \mathcal{B}_1$, since the same is true for each $\mathcal{F}$.

To show that $f_1, f_2 \in \mathcal{B}_1$ implies $f_1 - f_2 \in \mathcal{B}_1$, it suffices to show that $g \in \mathcal{B}_1 \Rightarrow -g \in \mathcal{B}_1$. To this end, first observe that if $\mathcal{F}$ fulfills the conditions (i), (ii), then also the set $-\mathcal{F} = \{-f; f \in \mathcal{F}\}$ does. Now let $g \in \mathcal{B}_2$ and $\mathcal{F}$ be an arbitrary set which fulfills (i) and (ii). Then $g \in -\mathcal{F}$, and thus $-g \in \mathcal{F}$. Since we have shown that $-g$ is contained in each set of functions fulfilling (i) and (ii), we get that $-g \in \mathcal{B}_2$.

Now from

$$f = \lim_{n\to\infty} f_n \Leftrightarrow f = f_1 + \sum_{n=1}^\infty (f_{n+1} - f_n)$$

we get that $\mathcal{B}_1$ is closed under pointwise limits. So this shows that $\mathcal{B} \subseteq \mathcal{B}_1$.

On the other hand, $\mathcal{B}$ fulfills the conditions (i) and (ii), hence $\mathcal{B}_1 \subseteq \mathcal{B}$. $\square$

Solution using transfinite induction. We have $\mathcal{B} = \bigcup_{\alpha\in\mathrm{On}} \mathcal{B}^\alpha$. (Where On denotes the class of all ordinal numbers.)

Let $\mathcal{A}^\triangle$ be the set of all functions of the form $f = \sum\limits_{n=1}^\infty f_n$, where each f_n belongs to $\mathcal{A}$ and the series $\sum f_n$ converges pointwise. (Here $\mathcal{A}$ can be any set of functions from $\mathbb{R}$ to $\mathbb{R}$.)

[29] In this book, the ordinals are introduced only in the next chapter and even there only ordinals less then ω_1 are using (in a disguise of well-ordered subsets of $\mathbb{Q}$). So the authors could not have used this argument, or at least not at this point.

Then we also have $\mathcal{B}_1 = \bigcup_{\alpha\in\mathrm{On}} \mathcal{C}^\alpha$, where $\mathcal{C}^0 = \mathcal{C}$, $\mathcal{C}^{\alpha+1} = (\mathcal{C}^\alpha)^\triangle$, and $\mathcal{C}^\alpha = \bigcup_{\beta<\alpha} \mathcal{C}^\beta$ if α is a limit ordinal.

To prove that $\mathcal{B}_1 = \mathcal{B}$ it suffices to show that $\mathcal{A}^* = \mathcal{A}^\triangle$ for any set of functions $\mathcal{A}$, which is group under (pointwise) addition; and to prove that if $(\mathcal{A}, +)$ is a group then also $(\mathcal{A}^*, +)$, $(\mathcal{A}^\triangle, +)$ are groups. $\square$

Exercise 16.I Consider all sets $\mathcal{F}$ of functions $\mathbb{R} \to \mathbb{R}$ for which
(i) $\mathcal{C} \subset \mathcal{F}$;
(ii) if $f_1, f_2, \dots \in \mathcal{F}$ is a monotone sequence of functions converging to f, then $f \in \mathcal{F}$.
Show that the smallest among these sets is $\mathcal{B}$. (Hint. Let $\mathcal{B}_2$ be the intersection of all these sets $\mathcal{F}$. Show that, if $f, g \in \mathcal{B}_2$, then $f \vee g \in \mathcal{B}_2$. Now use the formula $\lim\limits_{n\to\infty} f_n = \lim\limits_{p\to\infty} \lim\limits_{q\to\infty} (f_p \vee f_{p+1} \vee \dots \vee f_{p+q})$ to prove that $\mathcal{B}_2$ is closed under limits.)

If we show that $\mathcal{B}_2$ is closed under limits, then $\mathcal{B}_2$ is an L-set and $\mathcal{B} \subseteq \mathcal{B}_2$. On the other hand, $\mathcal{B}$ clearly fulfills the conditions (i) and (ii), so we also have the other inclusion $\mathcal{B}_2 \subseteq \mathcal{B}$.

The first part – without transfinite induction. Let us call a set $\mathcal{F}$ of real functions a $\mathrm{L}^\uparrow$-set if it fulfills condition (i) and (ii). The set $\mathcal{B}_2$ is the intersection of all $\mathrm{L}^\uparrow$-sets. We want to show that $\mathcal{B}_2$ is closed under $\vee$.

- Fix a continuous function f and define $M = \{g\colon \mathbb{R} \to \mathbb{R}; f \vee g \in \mathcal{B}_2\}$ We want to check whether M is an $\mathrm{L}^\uparrow$-set.
  We clearly have $\mathcal{C} \subseteq M$. Also if $g_n \in M$, $g_n \nearrow g$ then
  $$f \vee g = f \vee (\lim_{n\to\infty} g_n) = \lim_{n\to\infty} (f \vee g_n)$$
  and $(f \vee g_n)$ is a monotone sequences of functions. Hence $f \vee g \in \mathcal{B}_2$.
  So we have shown that M is an $\mathrm{L}^\uparrow$-set and thus $\mathcal{B}_2 \subseteq M$. This shows that $f \in \mathcal{C}$, $g \in \mathcal{B}_2 \Rightarrow f \vee g \in \mathcal{B}_2$.
- Now fix a function $g \in \mathcal{B}_2$ and define $M = \{f\colon \mathbb{R} \to \mathbb{R}; f \vee g \in \mathcal{B}_2\}$. We have shown $\mathcal{C} \subseteq N$ in the first part. Again, if $\lim\limits_{n\to\infty} f_n = f$ and f_n is a monotone sequence of functions from N, then we also have
  $$f \vee g = (\lim_{n\to\infty} f_n) \vee g = \lim_{n\to\infty} (f_n \vee g)$$
  and the sequence $(f_n \vee g)$ is also monotone. This shows that $f \vee g \in N$.
  So we have shown that N is an $\mathrm{L}^\uparrow$-set. This implies that $\mathcal{B}_2 \subseteq N$.

The second part of the preceding argument in fact says that $f, g \in \mathcal{B}_2 \Rightarrow f \vee g \in \mathcal{B}_2$. $\square$

The first part – using transfinite induction. Let $\mathcal{B}^\triangle$ the set of all limits of non-decreasing sequences of functions from $\mathcal{B}$.
Let $\mathcal{B}^\triangledown$ the set of all limits of non-increasing sequences of functions from $\mathcal{B}$.

We can inductively define $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_{\beta+1} = \mathcal{C}_\beta^{\triangle\triangledown}$ and $\mathcal{C}_\alpha = \bigcup\limits_{\beta<\alpha} \mathcal{C}_\beta$ for limit ordinal α.

Then we get $\mathcal{B}_2 = \bigcup_{\alpha \in \mathrm{On}} C_\alpha = \mathcal{C}_{\omega_1}$. It is also clear that $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$ for $\alpha \leq \beta$.

By induction we can show that each $\mathcal{C}_\alpha$ is closed under maxima, i.e., if $f, g \in \mathcal{C}_\alpha$ then also $f \vee g \in \mathcal{C}_\alpha$.

1° For $\alpha = 0$ we have that $f \vee g$ is continuous if both f and g are continuous.

2° It suffices to show that: a) If $\mathcal{B}$ is closed under maxima, then so is $\mathcal{B}^\triangle$.
b) If $\mathcal{B}$ is closed under maxima, then so is $\mathcal{B}^\triangledown$.

a)Suppose that $f_n \nearrow f$ and $g_n \nearrow g$, where $f_n, g_n \in \mathcal{B}$. Then $f \vee g = (\lim_{n\to\infty} f_n) \vee (\lim_{n\to\infty} g_n) = \lim_{n\to\infty}(f_n \vee g_n)$. This is a non-decreasing sequence of functions. So we get that $f \vee g \in \mathcal{B}^\triangle$.
b) The proof for $\mathcal{B}^\triangledown$ is similar.

3° If α is a limit ordinal and $f, g \in \mathcal{C}_\alpha$, then there exists a $\beta < \alpha$ such that $f, g \in \mathcal{C}_\beta$. Then we have $f \vee g \in \mathcal{C}_\beta \subseteq \mathcal{C}_\alpha$. □

The second part. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$. (I.e., the limit exists for each $x \in \mathbb{R}$.) This implies, in particular, that for each $x \in \mathbb{R}$ the sequence $(f_n(x))$ is bounded.

The sequence $f_p \vee f_{p+1} \vee \cdots \vee f_q$, $q \geq p$, is a monotone sequence. Since the values at each point are bounded, there exists a limit $F_p = \lim_{q\to\infty}(f_p \vee f_{p+1} \vee \cdots \vee f_q)$. Note that $\inf_{n\in\mathbb{N}} f_n(x) \leq F_p(x) \leq \sup_{n\in\mathbb{N}} f_n(x)$, so the sequence $(F_p(x))_{p\in\mathbb{N}}$ is also bounded for each $x \in \mathbb{R}$. We also have $F_p(x) \geq F_{p+1}(x)$. So the sequence $(F_p(x))_{p\in\mathbb{N}}$ is also monotone and it has a limit $L(x)$.

We clearly have $F_p(x) \geq f_p(x)$ and thus $L(x) \geq f(x)$.

Now let us fix some $x \in \mathbb{R}$ and choose an $\varepsilon > 0$. There exists p_0 such that $f_n(x) \leq f(x) + \varepsilon$ whenever $n \geq p_0$. This implies that $F_p(x) \leq f(x) + \varepsilon$ for each $p \geq p_0$ and thus $L(x) \leq f(x) + \varepsilon$. Since this is true for arbitrary $\varepsilon > 0$, we get $L(x) \leq f(x)$. □

After seeing this exercise, one might ask whether we would get all Borel functions if we used only non-decreasing pointwise convergent sequences of functions (instead of all monotone sequences, which include both non-decreasing and non-increasing sequences.) If $f_n(x)$ is a non-decreasing sequence and $f(x) = \lim_{n\to\infty} f_n(x)$, then we have $f(x) = \sup\{f_n(x); n \in \mathbb{N}\}$. From Theorem 10.3 we know that $\mathcal{C}^+$ is closed under arbitrary suprema.

So if we start with continuous functions and use only non-decreasing sequences, we will only obtain lower semicontinuous functions. Any upper semicontinuous function, which is not lower semicontinuous, is an example of a Borel function (in fact Baire-class-one function), which cannot be obtained in this way.

Exercise 16.J Consider all sets $\mathcal{F}$ of functions $\mathbb{R} \to \mathbb{R}$ for which
(i) $\mathcal{C} \subset \mathcal{F}$;
(ii) if $f_1, f_2, \cdots \in \mathcal{F}$ and the series $\sum |f_n|$ converges, then $f = \sum f_n \in \mathcal{F}$.
Show that the smallest of these sets is $\mathcal{B}$. (Hint. Let $\mathcal{B}_3$ be the intersection of all these sets $\mathcal{F}$. Show that $\mathcal{B}_3$ is a group under addition, and has the properties (i) and (ii) mentioned in the preceding exercise.)

Exercise 16.K. Let $f_1, f_2, \dots$ be Borel measurable functions on $\mathbb{R}$. Show that the following sets are Borel measurable.
(i) The set of all points x for which the sequence $f_1(x), f_2(x), \dots$ is increasing.
(ii) The set of all points x for which the sequence $f_1(x), f_2(x), \dots$ is bounded.
(iii) The set of all points x for which $\sum_{n=1}^{\infty} |f_n(x)| < \infty$.
(iv) The set of all points x for which $\lim\limits_{n\to\infty} f_n(x) = 0$.
(v) The set of all points x for which there exist infinitely many values of n with $f_n(x) > 0$.

Observation. If $f, g \in \mathcal{B}$ then the set $\{x \in \mathbb{R}; f(x) = g(x)\}$ belongs to Ω. (Since $f - g \in \mathcal{B}$, and this set is precisely $\{x \in \mathbb{R}; f(x) - g(x) = 0\}$.)

Solution. (i) We have $A_k = \{x \in \mathbb{R}; f_k(x) \le f_{k+1}(x)\} = \{x \in \mathbb{R}; f_k = f_k \wedge f_{k+1}\} \in \Omega$. The set of all points where the sequence is increasing is simply $\bigcap\limits_{k=1}^{\infty} A_k$.

(ii) Let $B_k = \{x \in \mathbb{R}; (\forall n \in \mathbb{N}) |f_n(x)| \le k\} = \bigcap_{n\in\mathbb{N}} \{x \in \mathbb{R}; |f_n(x)| \le k\}$. This set belongs to Ω. The set of all points where the sequence is bounded is precisely the set $\bigcup\limits_{k=1}^{\infty} B_k$.

(iii) Denote $g_n(x) = \sum_{k=1}^{n} |f_k(x)|$. Each function g_n is a Borel function. The set in this part is precisely the set of points where the sequence (g_n) is bounded; so the claim follows from (ii).

(iv) $\{x \in \mathbb{R}; \lim\limits_{n\to\infty} f_n(x) = 0\} = \{x \in \mathbb{R}; (\forall \varepsilon \in \mathbb{Q})(\exists n_0 \in \mathbb{N})(\forall n \ge n_0) |f_n(x)| < \varepsilon\} = \bigcap\limits_{\varepsilon\in\mathbb{Q}} \bigcup\limits_{n_0\in\mathbb{N}} \bigcap\limits_{n=n_0}^{\infty} f_n^{-1}((-\varepsilon, \varepsilon))$

(v) $\{x \in \mathbb{R}; (\exists^{\infty} n \in \mathbb{N}) f_n(x) > 0\} = \{x \in \mathbb{R}; (\forall n_0 \in \mathbb{N})(\exists n \ge n_0) f_n(x) > 0\} = \bigcap\limits_{n_0\in\mathbb{N}} \bigcup\limits_{n=n_0}^{\infty} f^{-1}((0, \infty))$ □

Exercise 16.L. Let $X_1, X_2, \dots$ be Borel sets whose union is $\mathbb{R}$. Suppose that $f_1, f_2, \dots \in \mathcal{B}$ and that $f\colon \mathbb{R} \to \mathbb{R}$ is such that for each n,

$$f(x) = f_n(x) \qquad (x \in X_n)$$

Then $f \in \mathcal{B}$.

Solution. Clearly, if such function exists then the functions f_n and f_m must have the same values on the set $X_n \cap X_m$.

For any $E \in \Omega$ we have

$$f^{-1}(E) = \bigcup_{n\in\mathbb{N}} \left(f_n^{-1}(E) \cap X_n\right).$$

Since each set $f_n^{-1}(E) \cap X_n$ belongs to Ω, their countable union $f^{-1}(E)$ is also in Ω. □

Exercise 16.M. Borel sets have property of Baire. For subsets A and B of $\mathbb{R}$ define $A \sim B$ if both $A \setminus B$ and $B \setminus A$ are meagre.
(i) $\sim$ is an equivalence relation. If $A \sim B$, then $\mathbb{R} \setminus A \sim \mathbb{R} \setminus B$. If $A_1, A_2, \dots$ and $B_1, B_2, \dots$ are subsets of $\mathbb{R}$ with $A_n \sim B_n$ for all n,then we have $\bigcup_n A_n \sim \bigcup_n B_n$ and $\bigcap_n A_n \sim \bigcap_n B_n$.
(ii) Let $V \subset \mathbb{R}$ and let U be the interior of $\overline{V}$. Show that $V \setminus U$ is meagre and that $U \setminus W$ is meagre for every open set W that contains V. Deduce that $V \sim U$ if V is a G_δ.
(iii) Now let $\mathcal{W}$ be the collection of all subsets A of $\mathbb{R}$ for which there exists an open set U with $A \sim U$. Prove that $\mathcal{W}$ has the following properties.[30]
(a) $\mathcal{W}$ contains all open sets and all meager sets.
(b) If $A \in \mathcal{W}$, then $\mathbb{R} \setminus A \in \mathcal{W}$.
(c) If $A_1, A_2, \dots \in \mathcal{W}$, then $\bigcup_n A_n \in \mathcal{W}$ and $\bigcap_n A_n \in \mathcal{W}$.
(d) Use Exercise 16.G to show that $\mathcal{W} \supset \Omega$.

Thus, a Borel set is, except for a meagre set, equal to an open set. (i) $\sim$ *is an equivalence relation.* Is is clear that $\sim$ is reflexive and symmetric. To show transitivity it suffices to notice that $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$.
$A \sim B \Rightarrow \mathbb{R} \setminus A \sim \mathbb{R} \setminus B$ is clear from $(\mathbb{R} \setminus A) \setminus (\mathbb{R} \setminus B) = B \setminus A$.
$A_n \sim B_n \Rightarrow \bigcup_n A_n \sim \bigcup_n B_n$ follows from $(\bigcup_n A_n) \setminus (\bigcup_k B_k) = \bigcup_n (A_n \setminus \bigcup_k B_k) \subseteq \bigcup_n (A_n \setminus B_n)$.
$A_n \sim B_n \Rightarrow \bigcap_n A_n \sim \bigcap_n B_n$ follows from $(\bigcap_n A_n) \setminus (\bigcap_k B_k) = \bigcup_k ((\bigcap_n A_n) \setminus B_k) \subseteq \bigcup_k (A_k \setminus B_k)$.
A different argument: $A_n \sim B_n \Rightarrow \mathbb{R} \setminus A_n \sim \mathbb{R} \setminus B_n \Rightarrow \bigcup_n (\mathbb{R} \setminus A_n) \sim \bigcup_n (\mathbb{R} \setminus B_n)$ $\Rightarrow \mathbb{R} \setminus (\bigcap_n A_n) \sim \mathbb{R} \setminus (\bigcap_n B_n) \Rightarrow \bigcap_n A_n \sim \bigcap_n B_n$

(ii) Let $V \subseteq \mathbb{R}$ and $U = \operatorname{Int} \overline{V}$.

$\overline{V} \setminus U$ *is nowhere dense.* The set $\overline{V} \setminus U$ is closed, so it suffices to show that it has empty interior.

Suppose that O is an open set such that $O \subseteq \overline{V} \setminus U$. We have:
$O \subseteq \overline{V} \Rightarrow O \subseteq \operatorname{Int} \overline{V} \Rightarrow O \subseteq U$
If we have both $O \subseteq \overline{V} \setminus U$ and $O \subseteq U$, then $O = \emptyset$.

If W is open and $W \supseteq V$, then $U \setminus W$ is meager. An open subset of $\mathbb{R}$ is an F_σ set (Theorem 6.2(iv)). The set $U \setminus W = U \cap (\mathbb{R} \setminus W)$ is also an F_σ set.

According to Exercise 6.I, an F_σ-set is meagre if and only if its interior is empty. So it suffices to show that $\operatorname{Int}(U \setminus W) = \emptyset$.

Suppose that there is a non-empty open set $O \subseteq U \setminus W$. Thus we have $x \in O \subseteq U \setminus W \subseteq \overline{V} \setminus W \subseteq \overline{V} \setminus V$. But the fact that O is an open neighborhood of x such that $O \cap V = \emptyset$ contradicts the assumption that $x \in \overline{V}$.

If V is G_δ, then $V \sim U$. We have $U = \operatorname{Int} \overline{V}$ and $V = \bigcap W_n$, where each W_n is open.
The set $V \setminus U \subseteq \overline{V} \setminus U$ is nowhere dense.
The set $U \setminus V = U \setminus (\bigcap_n W_n) = \bigcup_n (U \setminus W_n)$ is meager.

(iii) (a) It is clear directly from the definition that all open sets and all meager sets belong to $\mathcal{W}$.

[30] Sets from $\mathcal{W}$ are called sets with the *Baire property*, see [Ke, Section 8.F]. They are the smallest σ-algebra which contains all open and all meager sets, [Ke, Proposition 8.22].

(b) If $A \sim U$, then $\mathbb{R} \setminus A \sim \mathbb{R} \setminus U$. The set $\mathbb{R} \setminus U$ is closed and hence it is G_δ. From (ii) we get that any G_δ set belongs to $\mathcal{W}$. (Alternatively: The set V is closed, we have shown in (ii) that $V \setminus \operatorname{Int} V$ is meager, i.e., that $V \sim \operatorname{Int} V$.)

(c) If $A_n \sim U_n$ then $\bigcup_n A_n \sim \bigcup_n U_n$ and $\bigcup_n U_n$ is an open set. Similarly, $\bigcap_n A_n \sim \bigcap_n U_n$ and $\bigcap_n U_n$ is a G_δ-set.

(d) $\mathcal{W}$ contains all closed sets and it is closed under countable unions and intersections. Therefore $\Omega \subseteq \mathcal{W}$.

Exercise 16.N. For a sequence $a_1, a_2, \dots$ of real numbers we define a real number $\operatorname*{Lim}_{n\to\infty} a_n$ by

$$\operatorname*{Lim}_{n\to\infty} a_n := \begin{cases} \lim\limits_{n\to\infty} a_n & \text{if the sequence converges,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_1, f_2, \dots \in \mathcal{B}$. Then the set $E := \{x \in \mathbb{R};\ \lim\limits_{n\to\infty} f_n(x) \text{ exists}\}$ is a Borel set. (Hint. $x \in E$ if and only if the sequence $f_1(x), f_2(x), \dots$ is Cauchy.)

The function $x \mapsto \operatorname*{Lim}_{n\to\infty} f_n(x)$ is a Borel function. (Hint. Consider the sequence $f_1\zeta_E, f_2\zeta_E, \dots$)

Solution. $x \in E$ if and only if

$$(\forall \varepsilon > 0)(\exists n)(\forall p, q \geq n)|f_p(x) - f_q(x)| < \varepsilon.$$

For each $p, q \in \mathbb{N}$, $\varepsilon > 0$ the set $\{x \in \mathbb{R}; |f_p(x) - f_q(x)| < \varepsilon\}$ is a Borel set, since the function $x \mapsto |f_p(x) - f_q(x)|$ is Borel whenever f_p and f_q are. (Difference of two Borel functions is a Borel function. Composition of absolute value and a Borel function is a Borel function.)

Therefore

$$E = \bigcap_{\substack{\varepsilon>0 \\ ve\in\mathbb{Q}}} \bigcup_{n\in\mathbb{N}} \bigcap_{p=n}^{\infty} \bigcap_{q=n}^{\infty} \{x \in \mathbb{R}; |f_p(x) - f_q(x)| < \varepsilon\}$$

is also a Borel set.

Now we see that $\operatorname{Lim} f_n = \lim f_n \xi_E$, and $f_n \xi_E$ is a Borel function for each n. $\square$

Exercise 16.O. Let $X \subseteq \mathbb{R}$ be a Borel set and let $f\colon X \to \mathbb{R}$ be continuous. Define $f_X\colon \mathbb{R} \to \mathbb{R}$ by

$$f_X(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x \in \mathbb{R} \setminus X. \end{cases}$$

Then f_X is a Borel function.

Solution. Let U be an open subset of $\mathbb{R}$. Then $f^{-1}(U)$ is open in X, which means that it has the form $f^{-1}(U) = V \cap X$ for some open set V. This implies that $f_X^{-1}(U) \in \Omega$.

For the map f_X we have either $f_X^{-1}(U) = f^{-1}(U)$ or $f_X^{-1}(U) = f^{-1}(U) \cup (\mathbb{R} \setminus X)$, depending on whether or not $0 \in U$. In either of these two cases the set $f_X^{-1}(U)$ is a Borel set.

We have shown that $f_X^{-1}(U) \in \Omega$ for any open set U. Thus f_X is a Borel function by Theorem 16.7(δ). □

Exercise 16.P. Let $m, k \in \mathbb{N}$. Let Φ be a continuous map $\mathbb{R}^m \to \mathbb{R}^k$. Then $f \circ \Phi \in \mathcal{B}(\mathbb{R}^m)$ for every $f \in \mathcal{B}(\mathbb{R}^k)$.

Solution. Let $M = \{f \colon \mathbb{R}^m \to \mathbb{R}; f \circ \Phi \in \mathcal{B}(\mathbb{R}^m)\}$. It suffices to show that M is an L-set.

Clearly, if $f \in \mathcal{C}(\mathbb{R}^m)$, then $f \in M$. (Since $f \circ \Phi$ is continuous whenever f is continuous.)

Now let $f_k \in M$ and $f_k \to f$. Then

$$\lim_{k\to\infty} (f_k \circ \Phi) = f \circ \Phi.$$

Since $f_k \circ \Phi \in \mathcal{B}(\mathbb{R}^m)$ for each k, we get that also $f \circ \Phi \in \mathcal{B}(\mathbb{R}^m)$. This shows that also $f \in M$.

We have shown that f is an L-set. □

Exercise 16.Q. (i) If $f_1, \dots, f_m \in \mathcal{B}$, then the function

$$(x_1, \dots, x_m) \mapsto f_1(x_1) f_2(x_2) \dots f_m(x_m) \qquad ((x_1, \dots, x_m) \in \mathbb{R}^m)$$

is a Borel function on $\mathbb{R}^m$.
(ii) If $A_1 \dots A_m$ are Borel subsets of $\mathbb{R}$, then $A_1 \times \dots \times A_m$ is a Borel subset of $\mathbb{R}^m$.

Solution. (i) First we may notice that $\varphi_i(x_1, \dots, x_m) = f_i(x_i)$ is a Borel function, since $\varphi_i = f_i \circ p_i$ (and the i-th projection is continuous, so we can use Exercise 16.P.) The function in question is a product of such functions. Product of Borel functions is a Borel function.

(ii) It suffices to notice that $\chi_{A_1 \times \dots \times A_m} = \chi_{A_1} \chi_{A_2} \cdots \chi_{A_m}$. □

Exercise 16.R. Let I be an interval. Define the set $\mathcal{B}(I)$ of Borel functions $I \to \mathbb{R}$ by a simple modification of Definition 16.2 and 16.8.
(i) If $f \in \mathcal{B}$, then the restriction of f to I is a Borel function on I.
(ii) Conversely, if $f \in \mathcal{B}(I)$, then the function $f_I \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_I(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \in \mathbb{R} \setminus I \end{cases}$$

is an element of $\mathcal{B}$ (see Exercises 11.G and 16.O).
(iii) If $A \subset I$, then $\xi_A \in \mathcal{B}$ if and only if the restriction of ξ_A to I is an element of $\mathcal{B}(I)$. Thus, a subset of I is Borel if and only if it is 'Borel relative to I'.

Solution. We say that a set of functions from I to $\mathbb{R}$ is an L^I-set if it contains all continuous functions and it is closed under pointwise limits.

(i) It suffices to notice that $M = \{f\colon \mathbb{R} \to \mathbb{R}; f|_I \in \mathcal{B}(I)\}$ is an L^I-set.

(ii) It suffices to show that $M = \{f\colon I \to \mathbb{R}; f_I \in \mathcal{B}\}$ is an L^I-set

If f is continuous on I, then $f_I \in \mathcal{B}$ according to Exercise 16.O.

If $f_k \to f$, then also $(f_k)_I \to f_I$.

(iii) This follows from (i) and (ii). □

Exercise 16.S. Assuming that $\mathcal{B}^k \neq \mathcal{B}^{k+1}$ for all $k \in \mathbb{N}$, prove that $\mathcal{B}^\omega := \bigcup_k \mathcal{B}^k$ is not closed under limits a, and not even under uniform limits. (Define $\mathcal{B}^k(I)$ for intervals I and show that $\mathcal{B}^k(I) \neq \mathcal{B}^{k+1}(I)$ for all k and I (see Exercise 11.G). Choose pairwise disjoint intervals $I_1, I_2, \ldots$ and make $f_k \in \mathcal{B}^{k+1}(I_k) \setminus \mathcal{B}^k(I_k)$, $|f_k| \leq 1/k$.)

To solve this problem we will formulate a few auxiliary results.

Lemma. *Let $\varphi\colon \mathbb{R} \to I$ be a homeomorphism between $\mathbb{R}$ and an open interval I. Then $\varphi \circ f \in \mathcal{B}^k$ if and only if $f \in \mathcal{B}^k$.*

Proof. By induction on k.

1° For $k = 0$ we are talking about continuous function. Clearly $f \in \mathcal{C}$ implies $\varphi \circ f \in \mathcal{C}$. Similarly, $\varphi \circ f \in \mathcal{C}$ implies $f = \varphi^{-1} \circ \varphi \circ f \in \mathcal{C}$. (Composition of two continuous functions is continuous.)

2° Assume that the claim is true for $\mathcal{B}^k$.

Let $f \in \mathcal{B}^{k+1}$. Let $f_n \to f$, where each f_n belongs to $\mathcal{B}^k$. Then $\varphi \circ f_n \to \varphi \circ f$ (since φ is continuous), which implies that $\varphi \circ f \in \mathcal{B}^{k+1}$.

The implication $\varphi \circ f \in \mathcal{B}^{k+1} \Rightarrow f \in \mathcal{B}^{k+1}$ is shown in the same way, but using φ^{-1} instead of φ. □

This implies that if $\mathcal{B}^{k+1} \setminus \mathcal{B}^k \neq \emptyset$, then there exists also a bounded function in $\mathcal{B}_{k+1} \setminus \mathcal{B}_k$. (If we choose a function $f \in \mathcal{B}^{k+1} \setminus \mathcal{B}^k$, then the composition $\varphi \circ f$ is also in $\mathcal{B}^{k+1} \setminus \mathcal{B}^k$ and it is bounded.)

Lemma. *Let $\varphi\colon I \to \mathbb{R}$ be a homeomorphism between an open interval I and $\mathbb{R}$. Then we have $f \circ \varphi \in \mathcal{B}^k(I) \Leftrightarrow f \in \mathcal{B}^k$.*

Proof. By mathematical induction on k.

1° For $k = 0$ we have $\mathcal{B}^k(I) = \mathcal{C}(I)$.

If f is continuous, then $f \circ \varphi$ is continous. If $f \circ \varphi$ is continuous, then $f = (f \circ \varphi) \circ \varphi^{-1}$ is continuous.

2° Suppose that the claim is true for $\mathcal{B}^k$.

If $f \in \mathcal{B}^{k+1}$, then there is a sequence (f_n) of functions from $\mathcal{B}^k$ such that $f_n \to f$. Then we have $f_n \circ \varphi \to f \circ \varphi$. Since each $f_n \circ \varphi$ is in $\mathcal{B}^k(I)$, this shows that $f \circ \varphi \in \mathcal{B}^{k+1}(I)$.

Similarly, if $f \circ \varphi \in \mathcal{B}^{k+1}(I)$, then there are functions $g_n \in \mathcal{B}^k(I)$ such that $g_n \to f \circ \varphi$. Now we can repeat the same argument with $g_n \circ \varphi^{-1} \to (f \circ \varphi) \circ \varphi^{-1} = f$. $\square$

This lemma implies that if there is at least one (bounded) function in $\mathcal{B}^{k+1} \setminus \mathcal{B}^k$, then there is also a (bounded) function in $\mathcal{B}^{k+1}(I) \setminus \mathcal{B}^k(I)$.

Lemma. *If $f \in \mathcal{B}^k$, then $f|_I \in \mathcal{B}^k(I)$.*

Proof. 1° If f is continuous, then also the restriction $f|_I$ is continuous. $\square$

Lemma. *Let $k \geq 1$. Let*

$$f_I(x) = \begin{cases} f(x) & x \in I, \\ 0 & x \notin I. \end{cases}$$

Then $f \in \mathcal{B}^k(I) \Leftrightarrow f_I \in \mathcal{B}^k$

Proof. Induction on k.

1° For $k = 1$ this was shown in Exercise 11.G.

2° Assume that the claim is true for k, we will show that it holds also for $k + 1$.

$\boxed{\Rightarrow}$ Let $f \in \mathcal{B}^{k+1}(I)$ and $f_n \to f$, where $f_n \in \mathcal{B}^k(I)$. By the inductive hypothesis we have $(f_n)_I \in \mathcal{B}^k$. It is clear that $(f_n)_I \to f_I$, hence $f_I \in \mathcal{B}^{k+1}$.

$\boxed{\Leftarrow}$ Since $(f_I)|_I = f$, this follows from the preceding lemma. $\square$

From the preceding lemma we know that if $\mathcal{B}^{k+1}(I) \setminus \mathcal{B}^k(I)$ is non-empty, then there exists a function $f \in \mathcal{B}^{k+1} \setminus \mathcal{B}^k$ with $\operatorname{supp} f \subseteq I$. Moreover, f can be chosen to be bounded.

Solution. Let I_k be a sequence of disjoint intervals in $\mathbb{R}$.

Using the above results we have that if $\mathcal{B}^{k+1} \neq \mathcal{B}^k$, then there exists a function f_k such that $|f|_k \leq \frac{1}{k}$ and $\operatorname{supp} f \subseteq I_k$.

Then $f = \sum f_k$ exists and $f|_{I_k} = f_k$. Since $f|_I \notin \mathcal{B}^k(I)$, we get that $f \notin B^k$.

Hence $f \notin \bigcup\limits_{n<\omega} \mathcal{B}^k = \mathcal{B}^\omega$. $\square$

Exercise 16.T. Let ϕ be the function $\mathbb{R} \to [0, 1]$ that is periodic with period 1 and whose graph is sketched in Fig. 5.

(i) Let $\alpha_1, \alpha_2, \dots \in \{0, 1\}$ and $x := \sum\limits_{k=1}^{\infty} \alpha_k 3^{-k}$. Prove $\phi(3^{m-1}x) = \alpha_m$ $(m \in \mathbb{N})$.

(ii) For $x \in \mathbb{R}$ define

$$\psi_1(x) := \sum_{n=1}^{\infty} 2^{-n} \phi(3^{2n} x)$$

$$\psi_2(x) := \sum_{n=1}^{\infty} 2^{-n} \phi(3^{2n-1} x)$$

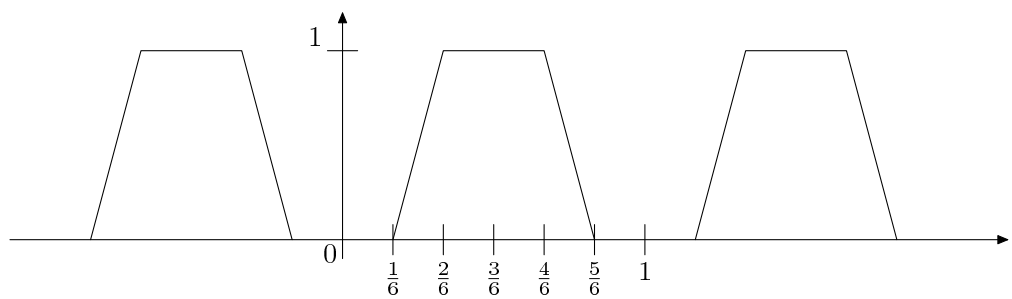

Figure 5: Function ϕ

Both ψ_1 and ψ_2 are continuous functions $\mathbb{R} \to [0,1]$. For all $y, z \in [0,1]$ there exists an $x \in [0,1]$ such that $\psi_1(x) = y$, $\psi_2(x) = z$. Thus, the formula

$$F(x) := (\psi_1(x), \psi_2(x)) \qquad (0 \le x \le 1)$$

yields a *continuous surjection* F of $[0,1]$ onto the square $[0,1] \times [0,1]$. (Such an F is called a *Peano curve.*)
(iii) (An extension of (ii).) Let λ be a bijection $\mathbb{N}^2 \to \mathbb{N}$, for example,

$$\lambda(m,n) = 2^m(2n-1) \qquad (m, n \in \mathbb{N}).$$

Define functions $\tau_1, \tau_2, \ldots$ or $\mathbb{R}$ by

$$\tau_m(x) := \sum_{n=1}^{\infty} 2^{-n} \phi(3^{\lambda(m,n)} x) \qquad (x \in \mathbb{R}).$$

Then each τ_m is a continuous map of $\mathbb{R}$ into $[0,1]$. If $y_1, y_2, \cdots \in [0,1]$, then there exists an $x \in [0,1]$ such that $\tau_1(x) = y_1$, $\tau_2(x) = y_2$, etc.

Solution. (i) We have $x = \sum\limits_{k=1}^{\infty} \alpha_k 3^{-k}$ where $\alpha_k \in \{0,1\}$, i.e., x is a number which has only 0's and 1's in its 3-adic expansion. Where can be such number located in the interval $\{0,1\}$?

If a first digit is 0 then the number x is between the numbers with 3-adic expansions $0.0000000\ldots$ and $0.011111111\ldots$; in the other words

$$\sum_{k=2}^{\infty} 0 \cdot 3^{-k} \le x \le \sum_{k=2}^{\infty} 1 \cdot 3^{-k}$$
$$0 \le x \le \frac{1}{3^2} \cdot \frac{1}{1-\frac{1}{3}} = \frac{3}{4 \cdot 3^2} = \frac{1}{6}$$

Similarly, if the first digit is 1 we get

$$\frac{1}{3} \le x \le \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

We see that for every number containing only zeroes and ones we have that $\phi(x)$ is equal to α_1.

Thus we already have that $\phi(3^{m-1}x) = \alpha_m$ holds for $m = 1$.

Now it suffices to notice that the 3-adic expansion of $3^{m-1}x = \alpha_1 \dots \alpha_{m-1}.\alpha_m\alpha_{m+1}\dots$, i.e., $3^m x = k + y$ where $k \in \mathbb{Z}$ and $y \in [0,1]$ is a number which has as the first digit the α_m and all its digits are zeroes and ones. We have already shown that the first digit is equal to the value of the function ϕ, so we get

$$\alpha_m = \phi(y) = \phi(3^{m-1}x).$$

Recapitulation. What we have so far: If we define $\phi_n(x) = \phi(3^n x)$, then we have defined functions ϕ_n, where $n = 1, 2, \dots$, such that:

- Each ϕ_n is continuous.
- $0 \le \phi(x) \le 1$
- If $x = \sum\limits_{k=1}^{\infty} \alpha_k 3^{-k}$, then $\phi_n(x) = \alpha_n$.

These properties are the only ones that will be needed to construct the function with the properties required in the remaining parts of this exercise.

(ii) Let us define

$$\psi_1(x) = \sum_{n=1}^{\infty} 2^{-n}\phi_{2n-1}(x)$$
$$\psi_2(x) = \sum_{n=1}^{\infty} 2^{-n}\phi_{2n}(x)$$

Both functions are sums of uniformly convergent functional series, therefore they are well defined and continuous. It is also clear that $0 \le \psi_i(x) \le \sum 2^{-n} = 1$.

If any numbers $y, z \in [0,1]$ are given, then we can rewrite them as

$$y = \sum_{k=1}^{\infty} \beta_k 2^{-k},$$
$$z = \sum_{k=1}^{\infty} \gamma_k 2^{-k},$$

where $\beta_k, \gamma_k \in \{0,1\}$.

By choosing $\alpha_{2k-1} = \beta_k$, $\alpha_{2k} = \gamma_k$ for $k = 1, 2, \dots$ and

$$x = \sum_{k=1}^{\infty} \alpha_k$$

we get $\psi_1(x) = y$ and $\psi_2(x) = z$.

(iii) Now let us define

$$\tau_m(x) = \sum_{n=1}^{\infty} 2^{-n}\phi_{\lambda(m,n)}(x).$$

Suppose we are given a sequence $y_1, y_2, \dots$ such that each $y_i \in [0,1]$. We can rewrite each of these numbers as

$$y_i = \sum_{k=1}^{\infty} \beta_{i,k} 2^{-k}.$$

If we choose $\alpha_{\lambda(m,n)} = \beta_{m,n}$ and $x = \sum\limits_{k=1\infty} \alpha_k 3^{-k}$ then

$$\tau_m(x) = \sum_{n=1}^{\infty} \phi_{\lambda(m,n)}(x) = \sum_{n=1}^{\infty} \beta_{m,n} 2^{-n} = y_m.$$

□

Exercise 16.V. A function $f\colon \mathbb{R} \to [0,1]$ that maps every subinterval of $\mathbb{R}$ onto all of $[0,1]$ cannot be of the first class of Baire, because functions of the fist class have continuity points.[31] However, we can now construct such a function that belongs to the second class. (Compare Exercise 16.A.) Let ϕ be as in Exercise 16.T(i). Define

$$f(x) := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(3^i x) \qquad (x \in \mathbb{R}).$$

(i) Show that $f \in \mathcal{B}^2$.
(ii) Clearly, f maps $\mathbb{R}$ onto $[0,1]$. We want to prove that f maps any subinterval of $\mathbb{R}$ onto $[0,1]$. First, observe that $f(3x) = f(x)$ and that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Deduce that $f(x + 3^{-p}) = f(x)$ for all $x \in \mathbb{R}$ and $p \in \{0,1,2,\dots\}$. Therefore it is enough to show that f maps $[0,1]$ onto $[0,1]$. Take $t \in [0,1]$. Find $\alpha_1, \alpha_2, \dots \in \{0,1\}$ such that

$$t = \limsup_{n\to\infty} \frac{1}{n}(\alpha_1 + \alpha_2 + \dots + \alpha_n)$$

(see Exercise 9.M). Show that $t = f(x)$, where $x := \sum_{k=1}^{\infty} \alpha_k 3^{-k}$.

Solution. (i) Since each

$$f_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \phi(3^i x)$$

sin continuous, the function $f = \limsup\limits_{n\to\infty} f_n$ is in $\mathcal{B}^2$ according to Exercise 16.C.

(ii) $\boxed{f(3x) = f(x)}$ We have

$$f_n(3x) - f_n(x) = \frac{\sum\limits_{i=1}^{n} \phi(3^i x) - \sum\limits_{i=0}^{n-1} \phi(3^i x)}{n} = \frac{\phi(3^n x) - \phi(x)}{n}.$$

[31] From Theorem 11.4 we know that the set of all points of discontinuity of f is a meagre F_σ.

Clearly $|f_n(3x) - f_n(x)| \le \frac{2}{n}$, and thus $\lim_{n\to\infty}(f_n(3x) - f_n(x)) = 0$. This implies

$$f(x) = \limsup_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(3x) = f(3x).$$

$\boxed{f(x+1) = f(x)}$ Again

$$f_n(x+1) = \frac{1}{n}\sum_{i=0}^{n-1}\phi(3^i x + 3^i) = \frac{1}{n}\sum_{i=0}^{n-1}\phi(3^i x) = f_n(x)$$

and thus $f(x+1) = f(x)$.

$\boxed{f(x+3^{-p}) = f(x)}$

$$f(x+3^{-p}) = f(3^p(x+3^{-p})) = f(3^p x + 1) = f(3^p x) = f(x).$$

Choice of α_n. Now it suffices to find for any given $t \in [0,1]$ a sequence of zeroes and ones such that

$$t = \lim_{n\to\infty}\frac{\alpha_1 + \cdots + \alpha_n}{n}.$$

We can simply define α_n by requiring that

$$\alpha_1 + \cdots + \alpha_n = \lfloor nt \rfloor,$$

i.e.,

$$\alpha_n = \lfloor nt \rfloor - \lfloor (n-1)t \rfloor.$$

Since $nt - (n-1)t = t \in [0,1]$, each α_n is either 0 or 1. It is clear that $\lim_{n\to\infty}\frac{\lfloor nt \rfloor}{n} = t$.

$\boxed{f(x) = t}$ Since we know from Exercise 16.T that $\phi(3^{i-1}x)$ returns α_i for $x = \sum_{k=1}^{\infty}\alpha_k 3^{-k}$ we get that

$$f(x) = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\phi(3^i x) = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_i = t.$$

□

5.17 Transfinite construction of the Borel functions

Exercise 17.E. (i) For $x \in \mathbb{Q}$ define

$$\phi(x) := \begin{cases} 1 - \frac{1}{2+2x} & \text{if } x \ge 0, \\ \frac{1}{2-2x} & \text{if } x < 0. \end{cases}$$

Then ϕ defines a strictly increasing bijection $\mathbb{Q} \to \mathbb{Q} \cap (0,1)$. Thus if D is a well-ordered subset of $\mathbb{Q}$, then $\phi[D]$ is a well-ordered subset of $\mathbb{Q} \cap (0,1)$.

Solution of (i). It suffices to notice that we get

$$y = 1 - \frac{1}{2(1+x)} \Rightarrow 2(1+x) = \frac{1}{1-y} \Rightarrow x = \frac{1}{2(1-y)} - 1$$

and for $1/2 < y < 1$ we get all values of x in the interval $(0, \infty)$.

Similarly we have

$$y = \frac{1}{2(1-x)} \Rightarrow 2(1-x) = \frac{1}{y} \Rightarrow x = 1 - \frac{1}{2y}$$

and for $0 < y \leq 1/2$ we get values of x in the interval $(-\infty, 0]$.

Therefore ϕ is a bijection between $\mathbb{Q}$ and $\mathbb{Q} \cap (0,1)$.

We also see that ϕ also gives us a bijection between $\mathbb{R} \setminus \mathbb{Q}$ and $(0,1) \setminus \mathbb{Q}$. $\square$

5.18 Analytic sets

Continuity of inverse of f. It can be shown that $f\colon \mathscr{N} \to \mathbb{R}$ defined by $f(a_1, a_2, \dots) = 2^{-a_1} + 2^{-a_1-a_2} + 2^{-a_1-a_2-a_3} + \dots$ is continuous.

When expressed in dyadic expansion, this function maps a sequence $a = (a_1, a_2, a_3, \dots)$ to a number where the dyadic expansions is composed of blocks of length a_i:

$$0.\underbrace{0001}_{a_1}\underbrace{001}_{a_2}\underbrace{00001}_{a_3}\dots$$

For example:

$$f(1,2,1,2,\dots) = 0.101101101\dots$$
$$f(1,1,1,1,\dots) = 0.111111\dots$$

TODO continuity, continuity of inverse (using dyadic expressions)

Exercise 18.B. Prove that the collection of all analytic sets has the cardinality of the continuum, so that there must exist nonanalytic sets. (Let $\mathscr{N}_1$ be the set of all elements of $\mathscr{N}$ that are eventually constant. $\mathscr{N}_1$ is countable, and a continuous function $f\colon \mathscr{N} \to \mathbb{R}$ is completely determined by its restriction to $\mathscr{N}_1$.)

This implies that we cannot show that there are analytic sets, which are not Borel, just using the cardinality. Existence of such sets is shown later in this chapter.

Exercise 18.C. Show that for a nonempty $A \subset \mathbb{R}$ the following are equivalent.
(α) A is analytic.
(β) There is a continuous surjection $\mathbb{R} \setminus \mathbb{Q} \to A$.
(γ) There is a continuous surjection $[0,1] \setminus \mathbb{Q} \to A$.
(For the proof of the implication (α) $\Rightarrow$ (β), note that Exercise 18.A provides a continuous map $[0,1] \setminus \mathbb{Q}$ whose image contains all but countably many elements of $\mathscr{N}$; it is easy to extend this map to a continuous surjection $\mathbb{R} \setminus \mathbb{Q} \to \mathscr{N}$.)

Proof. The function $g\colon (0,1] \to \mathscr{N}$ from Exercise 18.A is continuous at all irrational points and it is bijective and it is continuous at all irrational numbers. Thus we have a continuous map $g|_{[0,1]\setminus\mathbb{Q}}$ from $[0,1]\setminus\mathbb{Q}$ to $\mathscr{N}$. There are only countably many points which do not have any preimage in this map, namely the points of the set $g[(0,1]\cap\mathbb{Q}] = \{x_1, x_2, x_3, \dots\}$. We can obtain a continuous map $\mathbb{R}\setminus\mathbb{Q} \to \mathscr{N}$ by putting $g(x) = x_n$ for $x \in [n, n+1]\cap\mathbb{Q}$ and $g(x) = 0$ for $x < 0$. This shows that $(\alpha) \Rightarrow (\beta)$.

The map ϕ from Exercise 17.E yields a continuous bijection $\mathbb{R}\setminus\mathbb{Q} \to [0,1]\setminus\mathbb{Q}$. The inverse ϕ^{-1} is also continuous. So we see that $(\beta) \Leftrightarrow (\gamma)$.

We already know, that all Borel sets are analytic (Theorem 18.8). So the sets $\mathbb{R}\setminus\mathbb{Q}$ and $[0,1]\setminus\mathbb{Q}$ are analytic and thus we get $(\beta) \Rightarrow (\alpha)$ and $(\gamma) \Rightarrow (\alpha)$. $\square$

It is worth mentioning that the spaces $\mathbb{R}\setminus\mathbb{Q}$, $[0,1]\setminus\mathbb{Q}$ and $\mathscr{N}$ are, in fact, homeomorphic.

One possibility to construct such a homeomorphism is to use *continued fraction*. Every real number can be expressed in the form of continued fraction

$$[a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where a_0 is an integer and all remaining a_i's are positive integers.

Rational numbers are precisely the numbers for which there exists a finite continued fraction. For each rational numbers there are two expressions in the form of continued fractions, for example $\frac{1}{3} = \frac{1}{2+\frac{1}{1}}$.

For every irrational number there exists exactly one infinite continued fraction.

Therefore continued fractions give us a bijection between $\mathscr{N}$ (the set of all sequences of positive integers) and irrationals in the interval $(0,1)$. It can be shown that this bijection is in fact a homeomorphism. A homeomorphism between $\mathscr{N}$ and $\mathbb{R}\setminus\mathbb{Q}$ can be obtained in the same way.

However, to give a detailed proof that this is indeed a homeomorphism, we would have to mention at least basic properties of continues fractions.[32] A different proof, avoiding the use of continued fractions can be found in [M, Theorem 1.1]. (The same proof is given in [AB, Theorem 3.68].)

Exercise 18.D. A nonempty subset A of $\mathbb{R}$ is analytic if and only if A is the set of all values of some left continuous function on $(0,1]$. (Hint. Exercise 18.A yields a left continuous surjection $(0,1] \to \mathscr{N}$. On the other hand, if $f\colon (0,1] \to \mathbb{R}$ is left continuous, then by Theorem 7.7 the set X of all discontinuity points of f is countable. As a continuous image of a Borel set, $f[(0,1]\setminus X]$ is analytic.)

[32] See `http://math.stackexchange.com/questions/352547/baire-space-homeomorphic-to-irrationals`.

Exercise 18.E. Let $A_1, A_2, \dots$ be pairwise disjoint analytic sets. Then there exist pairwise disjoint Borel sets $E_1, E_2, \dots$ such that $A_n \subset E_n$ for each n.

Solution. We apply separation lemma to the sets A_1 and $\bigcup_{k=2}^{\infty} A_k$, then to A_2 and $E_1 \cup \left(\bigcup_{k=3}^{\infty} A_k\right)$, then to A_3 and $(E_1 \cup E_2) \cup \left(\bigcup_{k=4}^{\infty} A_k\right)$, etc. $\square$

Exercise 18.F. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and surjective. Let $A \subset \mathbb{R}$. Then A is Borel if and only if $f^{-1}(A)$ is Borel.

Solution. $\boxed{\Rightarrow}$ Continuous functions are Borel. The result follows now from Theorem 16.7.

$\boxed{\Leftarrow}$ If $f^{-1}(A)$ is Borel, then both $f^{-1}(A)$ and $\mathbb{R} \setminus f^{-1}(A)$ are analytic. Now, since f is surjective, we have

$$\begin{aligned} A &= f[f^{-1}(A)] \\ \mathbb{R} \setminus A &= f[f^{-1}(\mathbb{R} \setminus A)] = f[\mathbb{R} \setminus f^{-1}(A)] \end{aligned}$$

which means that both A and $\mathbb{R} \setminus A$ are analytic. Hence A is Borel. $\square$

Exercise 18.I. Let $f\colon \mathbb{R} \to \mathbb{R}$. Show that f is Borel measurable if and only if for every analytic set $X \subset \mathbb{R}$ the set $f^{-1}(X)$ is analytic. (Using the methods of the proof of Theorem 18.12 and observing that, for all $X \subset \mathbb{R}$, $f^{-1}(X) = \pi_1(\Gamma \cap R \times X)$ one shows that, if f is Borel and X is analytic, then $f^{-1}(X)$ is analytic. For the converse, use Theorem 18.10 to show that $f^{-1}(X)$ is Borel for every Borel set X.)

Solution. $\boxed{\Rightarrow}$ If f is Borel then the graph Γ of f is Borel (Theorem 18.12). If X is analytic, then so is $\mathbb{R} \times XC$. We have $f^{-1}(X) = \pi_1(\Gamma \cap \mathbb{R} \times X)$, hence $f^{-1}(X)$ is a continuous image of an analytic set.

$\boxed{\Leftarrow}$ Suppose that X is Borel. This implies that both X and $\mathbb{R} \setminus X$ are analytic. Now $f^{-1}(X)$ and $\mathbb{R} \setminus f^{-1}(X) = f^{-1}(\mathbb{R} \setminus X)$ are both analytic, which implies that $f^{-1}(X)$ is Borel. $\square$

Exercise 18.J. Prove the equivalence of the statements (α), (β), (γ), (ε) of the beginning of this section. (That we may add (β) to this list will follow from Theorem 18.15(ii).)

Solution. Let us recall first the conditions (α)–(ε):
(α): A is the image of a Borel set E under a Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$.
(β): A is the image of a G_δ-set E under a Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$.
(γ): A is the image of a G_δ-set E under a Borel measurable function $f\colon E \to \mathbb{R}$.
(δ): There exists a continuous map of $\mathbb{R} \setminus \mathbb{Q}$ onto A.
(ε) There exists a continuous map of $\mathscr{N}$ onto A.

We want to show that the above statements are equivalent for $A \neq \emptyset$. (With the exception of (β), which will be shown later.)

We have shown equivalence $(\delta) \Leftrightarrow (\varepsilon)$ in Exercise 18.C.

The implication $(\delta) \Rightarrow (\gamma)$ is clear.

$(\gamma) \Rightarrow (\alpha)$: In Exercise 16.O we have shown that the function

$$f_E(x) = \begin{cases} f(x) & x \in E, \\ 0 & \text{otherwise} \end{cases}$$

is a Borel measurable function if $f\colon E \to \mathbb{R}$ is continuous. With an obvious modification, where we replace 0 by an arbitrary (but fixed) element $a \in A$, we get a Borel measurable extension of the function f such that $f[\mathbb{R}] = A$.

$(\alpha) \Rightarrow (\varepsilon)$: This follows from Theorem 18.14. $\square$

Exercise 18.K. For a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ let E_f denote the set of points where f is differentiable. Prove the following theorem (due to Poprougenko). A subset X of $\mathbb{R}$ is analytic if and only if there exists an $f \in \mathcal{C}(R)$ with $X = \{f'(x); x \in E_f\}$. (The 'if' follows easily from Theorems 7.11 and 18.14. The proof of the converse is somewhat more involved. Let X be analytic, for simplicity assume $X \cap [-1,1] \neq \emptyset$. Use Exercise 18.D to obtain a left continuous $g\colon (0,\infty) \to \mathbb{R}$ that, for each $n \in \mathbb{N}$, maps $(0,n]$ onto $X \cap [-n,n]$. By Theorem 7.7 there exists an infinite sequence $x_1, x_2, \dots$ in $(0,\infty)$ such that every discontinuity point of g is an x_k. Extend g to a left continuous $h\colon \mathbb{R} \to \mathbb{R}$ with $h(x) = g(x_k)$ for all $x \in (-k, -k+1]$ $(k \in \mathbb{N})$. Show that h is Riemann integrable over every interval $[a,b]$: define $f(x) := \int_0^x h(t)\,\mathrm{d}t$ $(x \in \mathbb{R})$.

Solution. $\boxed{\Leftarrow}$ Theorem 7.11 says that, for a continuous function f, the set E_f is an $F_{\sigma\delta}$-set. Hence it is a Borel set and, consequently, an analytic set. Let us define[33]

$$d(x) = \limsup_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n} = \limsup_{n\to\infty} n(f(x+1/n) - f(x)).$$

The function $d(x)$ is in $\mathcal{B}^2$ according to Exercise 16.C. (Since $x \mapsto n(f(x+1/n) - f(x))$ is continuous for each n.) Moreover, if f is differentiable at x, then $f'(x) = d(x)$.

Now $X = \{f'(x); x \in E_f\} = \{d(x); x \in E_f\}$ is an image of a Borel set E_f under the Borel measurable function d. So it is analytic by Theorem 18.14.

$\boxed{\Rightarrow}$ Let X be an analytic set. From Exercise 18.D we have a left continuous surjective function[34] $g_n\colon (n-1,n] \to X \cap [-n,n]$. Combining these functions we get a left continuous surjection $g\colon (0,\infty) \to X$.

[33] This is similar to $D^+f(x)$, see Definition 15.4.

[34] Q: Why could not we use simply $g\colon (0,1] \to X$? A: Riemann integral is defined (in this book) only for bounded functions. So we need g to be bounded on each interval $[0,x]$, so that the expression $\int_0^x g(t)\,\mathrm{d}t$ makes sense. We will also use the fact g is locally bounded in the proof that $f(x)$ is continuous.

By Theorem 7.7 there are at most countably many discontinuity points x_k of the function $g(x_k)$. This implies that g is Riemann integrable. (A function such that the set of points of discontinuity is a null set is Riemann integrable, see Theorem 12.1.).

Now we define $f(x) = \int_0^x g(t)\,\mathrm{d}t$.

If x is a point of continuity of g, then we can show $f'(x) = g(x)$. Namely if $\delta > 0$ is such that $|y - x| < \delta \Rightarrow |g(y) - g(x)| < \varepsilon$ then

$$\frac{f(y) - f(x)}{y - x} \leq \frac{\int_x^y g(t)\,\mathrm{d}t}{y - x} \leq \frac{(g(x) + \varepsilon)|y - x|}{|y - x|} = g(x) + \varepsilon$$

and the same argument shows that this fraction is at least $f(x) - \varepsilon$. This shows that

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = g(x).$$

(This is basically Exercise 16.O.)

If x is not a point of continuity of g, then f may or may not be differentiable at x. But if it is differentiable, the same argument as above shows that $f'(x) = g(x)$. (Using that x is left-continuous and taking only $y - x$.)[35]

So at the moment we know that

$$X \setminus \{g(x_k); k \in \mathbb{N}\}\{f'(x); x \in E_f\} \subseteq X.$$

If we define $h(x) = g(x_k)$ for $x \in (-k, -k + 1]$, we get a function h such that $\{h'(x); x \in E_h\} = X$.

It only remains to show that $f(x)$ is continuous. To see this, just notice that if we choose some interval $(x - \delta, x + \delta)$ around x, then the function $g(x)$ is bounded on this interval. So let $|g(x)| \leq M$. Then we get

$$|f(y) - f(x)| = \left| \int_x^y g(t)\,\mathrm{d}t \right| \leq M|y - x|$$

for any $y \in (x - \delta, x + \delta)$. □

Proof of Theorem 18.15(ii). *It suffices to construct a continuous surjection* $G\colon \mathbb{R} \to [0, 1] \times \mathbb{R}$. Indeed, any analytic set can by obtained as $X = \pi_2(A)$, for some G_δ subset $A \subseteq \mathbb{R}^2$. Since G is continuous, the set $G^{-1}(A)$ is also G_δ. Since G is surjective, we have $G[G^{-1}(A)] = A$, which implies

$$\pi_2 \circ G[G^{-1}(A)] = \pi_2[A] = G.$$

Construction of continuous surjection $\mathbb{R} \mapsto \mathbb{R} \times \mathbb{R}$. In Exercise 16.T, a continuous surjection $F\colon [0, 1] \to [0, 1] \times [0, 1]$ was constructed. This can be easily modified to a continuous surjection $F_n\colon [0, 1] \to [-n, n] \times [-n, n]$.

[35] We have shown that $D_l f(x) = \lim\limits_{y \to x^-} \frac{f(y) - f(x)}{y - x} = g(x)$. The left derivative $D_l f(x)$ was defined in Section 2. If a function is differentiable, then $f'(x) = D_l f(x) = D_r f(x)$.

Using these maps we can map $[0,1]$ into $[-1,1]^2$, $[2,3]$ onto $[-2,2]^2$ etc. Then we can simply join these parts by a segment line.

$$G(x) = \begin{cases} F_n(t), & x = (2n-2)+t, n \in \mathbb{N}, t \in [0,1] \\ tF_n(1) + (1-t)F_{n+1}(t) & x = (2n-1)+t \\ F_1(0), & x < 0. \end{cases}$$

In this way he obtain a continuous function. It covers bigger and bigger squares, in the and we get $G[\mathbb{R}] = \mathbb{R}^2$.

The construction given in the book is rather similar. They construct a continuous surjection $\mathbb{R} \to [0,1] \times \mathbb{R}$. The main difference is that the function F in Exercise 16.T was chosen in a such way that we get $F_n(0) = F_n(1) = (0,0)$ for each n, so there is no needs to join the parts of the function by line segments. (If we choose the enlarging square in a such way, that they share a vertex.)

Proof of Lemma 18.18. The fact, that $\mathscr{N}^2 \setminus \Gamma$ is a union of squares, follows from the fact, that graph of a continuous function is closed.

Proof of 18.19. To see that $\Delta_{23} \cap W$ is analytic, it suffices to notice that Δ_{23} is closed. (This implies that $\Delta_{23} \cap W$ is closed, hence $\Delta_{23} \cap W$ is analytic.)

Since $\mathscr{N}^3 \cong \mathscr{N}$ are homeomorphic[36], properties of analytic sets in $\mathscr{N}^3$ and $\mathscr{N}$ are the same. And we already know that any closed subset of $\mathscr{N}$ is analytic.

6 Integration

6.19 The Lebesgue integral

Exercise 19.A. (i) The characteristic function of a bounded open set is in $\mathscr{L}$. (Hint. Use Theorem 10.6.)
(ii)The characteristic function of a bounded closed set is in $\mathscr{L}$.

Solution. (i) Let A be a bounded open set. We know from Theorem 10.2(i) that χ_A is lower semicontinuous. So by theorem 10.6 there exists a non-decreasing sequence f_n of continuous functions such that $f_n \nearrow \chi_A$. W.l.o.g. we can assume $f_n \geq 0$. (If needed, we can replace f_n by $f_n \vee 0$, which influences neither monotonicity of the sequence, nor the convergence to χ_A.) So we have $0 \leq f_n \leq \chi_A$.

Now if we set $\phi_1 = f_1$ and $\phi_{n+1} = f_{n+1} - f_n$, then $\sum\limits_{k=1}^{n} \phi_k = f_n$, hence $\sum\limits_{k=1}^{\infty} \phi_k = f$.

Also we have $0 \leq \sum\limits_{i=1}^{n} \int |\phi_i| = \sum\limits_{i=1}^{n} \int \phi_i = \int f_n \leq \int \chi_A < +\infty$. (Since the set A is bounded.)

[36] We have $\mathscr{N}^3 = (\mathbb{N}^{\aleph_0})^3 = \mathbb{N}^{3\aleph_0}$ and $\mathscr{N} = \mathbb{N}^{\aleph_0}$.

(ii) If A is a bounded closed set then there exists some interval $I \supseteq A$ and a non-increasing sequence f_n of continuous functions, which are zero outside I, such that $f_n \swarrow \chi_A$. Again, we can put $\phi_1 = f_1$ and $\phi_{n+1} = f_{n+1} - f_n$.

Again, $\sum\limits_{i=1}^{\infty} \phi_i = f$.

We also have $\sum\limits_{i=1}^{n} |\phi_i| = \phi_1 + \sum\limits_{i=2}^{n} (f_{n-1} - f_n) = f_1 + f_2 - f_n$ which means that

$$\sum_{i=1}^{n} \int |\phi_i| \leq \int f_1 + f_2 \leq 2 \int \chi_I < +\infty.$$

□

Exercise 19.B. Prove or disprove the following statement. If $f \in \mathscr{L}$, $f \geq 0$, then there exist $\phi_1, \phi_2, \dots \in \mathscr{C}_c$ such that $\phi_i \geq 0$ for all i, $\sum\limits_{i=1}^{\infty} \int \phi_i$ is finite, $\sum\limits_{i=1}^{\infty} \phi_i = f$ a.e.[37]

Solution. Let us start by a simple observation on functions which can be obtained as $f = \sum \phi_i$, where each ϕ_i is continuous and non-negative. Then the set $\{x \in \mathbb{R}; f(x) > 0\}$ is open. (This can be seen from $\{x \in X; f(x) > 0\} = \bigcup\limits_{n=1}^{\infty} \{x \in \mathbb{R}; \sum\limits_{i=1}^{n} \phi_i(x) > 0\}$, so this is a union of open sets, since each $\sum\limits_{i=1}^{n} \phi_i$ is a continuous function. Or we can use the fact that f is lower semicontinuous and the characterization of lower semicontinuous functions; Theorem 10.2.)

Let us now consider the function $g = \chi_A$ for some bounded set A.

Suppose that $g \overset{\text{a.e.}}{=} f$ for some $f = \sum \phi_i$ with $\phi_i \geq 0$, $\phi_i \in \mathscr{C}_c$. The set $U = \{x \in \mathbb{R}; f(x) > 0\}$ is an open set. Since g and f are equal almost everywhere $\{x \in \mathbb{R}; g(x) > 0\} \triangle \{x \in \mathbb{R}; f(x) > 0\}$ is a nullset. I.e., $A \triangle U$ is a nullset.

If A is a closed bounded set, then $\chi_A \in \mathscr{L}$ by Exercise 16.A(ii). So it suffices to find a closed bounded set A such that there is no open set fulfilling $m(U \triangle A) = 0$.

Let A be a bounded closed nowhere dense set, which is not a nullset.[38] The complement $O = \mathbb{R} \setminus A$ is a dense open set, which does not have full measure.

Now for any non-empty open subset $U \subseteq \mathbb{R}$ we get that $U \cap O$ is non-empty open set. A non-empty open set is not a nullset. This means that $U \triangle A \supseteq U \cap (\mathbb{R} \setminus A) = U \cap O$ is not a nullset.

---

[37] Note that every function of the form $\sum\limits_{i=1}^{\infty} \phi_i$, where ϕ_i's are continuous, is lower semicontinuous (Theorem 10.3). So this is close to the question: *Is every Lebesgue integrable function equal a.e. to some lower semicontinuous function?*

[38] We have seen a construction of such set in Chapter 5 using enumeration $\{r_i; i = 1, 2, \dots\}$ of rationals and taking $O = \bigcup\limits_{i=1}^{\infty} \left(r_i - \frac{1}{2^i}, r_i + \frac{1}{2^i}\right)$. It is easy to modify this construction to get a bounded set. There are also other constructions, for example fat Cantor set.

So $m(U \triangle A)$, where U is open, is only possible for $U = \emptyset$. But this would imply that A is a nullset, which is a contradiction.

Therefore χ_A is an example of a function such that $\chi_A \in \mathscr{L}$, but χ_A cannot be obtained as a sum of non-negative ϕ_i's with the above properties. $\square$

Exercise 19.D. Let $f \in \mathscr{L}$.
(i) If $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and $g \geq 0$, then $f \wedge g \in \mathscr{L}$.
(ii) If $h\colon \mathbb{R} \to \mathbb{R}$ is bounded and continuous, then $fh \in \mathscr{L}$.

Proof. Since $f \in \mathscr{L}$ is continuous, there exist functions $\psi \in \mathscr{C}_c$ such that $\psi \to f$ and $\sum \int |\psi_{i+1} - \psi_i| < +\infty$.

(i) Let us define $\varphi_i := \psi_i \wedge g$. Each φ_i belongs to $\mathbb{C}_c$. (It is continuous and it is zero whenever $\varphi_i(x) = 0$.)

To show that $\sum \int |\varphi_{i+1} - \varphi_i| < +\infty$, it suffices to show that

$$|\varphi_{i+1} - \varphi_i| \leq |\psi_{i+1} - \psi_i|.$$

Using the identity $2(f \wedge g) = f + g - |f - g|$. We have

$$|\varphi_{i+1} - \varphi_i| = |\psi_{i+1} \wedge g - \psi_i \wedge g| = \frac{1}{2}|\psi_{i+1} + g - |\psi_{i+1} - g| - (\psi_i + g - |\psi_i - g|)| =$$
$$= \frac{1}{2}|\psi_{i+1} - \psi_i + |\psi_i - g| - |\psi_{i+1} - g|| \leq \frac{1}{2}|\psi_{i+1} - \psi_i| + \frac{1}{2}||\psi_i - g| - |\psi_{i+1} - g|| \leq$$
$$\leq \frac{1}{2}|\psi_{i+1} - \psi_i| + \frac{1}{2}|(\psi_i - g) - (\psi_{i+1} - g)| = \frac{1}{2}|\psi_{i+1} - \psi_i| + \frac{1}{2}|\psi_i - \psi_{i+1}| = |\psi_{i+1} - \psi_i|$$

By considering various cases. If $g(x)$ is less than both $\psi_i(x)$ and $\psi_{i+1}(x)$, then $\varphi_{i+1}(x) - \varphi_i(x) = \psi_{i+1}(x) - \psi_i(x)$.

If $g(x)$ is greater than both $\psi_i(x)$ and $\psi_{i+1}(x)$, then $\varphi_{i+1}(x) - \varphi_i(x) = 0$.

So the only remaining case is that $g(x)$ is between $\psi_i(x)$ and $\psi_{i+1}(x)$. Then one of the values $\varphi_j(x)$, where $j \in \{i, i+1\}$ is equal to the corresponding ψ_j

$f \wedge g$ is the limit of φ_i. We also have

$$\lim_{i \to \infty} (\psi_i(x) \wedge g(x)) = (\lim_{i \to \infty} \psi_i(x)) \wedge g(x) = f(x) \wedge g(x).$$

(This follows from the fact that $\wedge\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Or we can use the identity $2(f \wedge g) = f + g - |f - g|$ again.) $\square$

Exercise 19.G. Reconsider Exercise 19.C in the light of the above theorems.

Exercise 19.H. Let $g\colon \mathbb{R} \to \mathbb{R}$ be bounded, increasing and differentiable. Set $A := \lim_{x \to -\infty} g(x)$, $B := \lim_{x \to \infty} g(x)$. Using Fatou's theorem, show that g' is Lebesgue integrable and that $\int g' \leq B - A$. (Yes, they are equal, but the proof is complicated. See Section 21.)

Solution. Let us define

$$g_n(x) = \begin{cases} \frac{g\left(x+\frac{1}{n}\right)}{\frac{1}{n}} & x \in [-n, n], \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\lim\limits_{n\to\infty} g_n = g'$.

Since each g_n is continuous, it is also Riemann integrable on $[-n, n]$ and thus $g_n \in \mathscr{L}$.

Since g is increasing, we get $g_n \geq 0$.

We want also to show that $\int g_n \leq B - A$. (This shows that the sequence $\int g_1, \int g_2, \ldots$ is bounded and this also yields the estimate for the limit g'.) To show this we divide the interval $[-n, n]$ into subintervals of length $\frac{1}{n}$. We have

$$\int g_n(x) = n \int_{-n}^{n} (g(x + 1/n) - g(x)) =$$
$$= n \left(\int_{-n}^{-n+1/n} g(x + 1/n) - \int_{-n}^{-n+1/n} g(x) + \cdots + \int_{n}^{n+1/n} g(x + 1/n) - \int_{n}^{n+1/n} g(x) \right)$$

If we use that fact that $\int\limits_{a}^{a+1/n} g(x + 1/n) = \int\limits_{a+1/n}^{a+2/n} g(x)$, we get

$$\int g_n(x) = n \left(-\int_{-n}^{-n+1/n} g + \int_{-n+1/n}^{-n+2/n} g - \int_{-n+1/n}^{-n+2/n} g + \int_{-n+2/n}^{-n+3/n} g + \cdots + -\int_{n}^{n+1/n} g + \int_{n+1/n}^{n+2/n} g \right) =$$
$$= n \left(\int_{n+1/n}^{n+2/n} g - \int_{-n}^{-n+1/n} g \right) \leq n \cdot \frac{1}{n}(B - A) = B - A.$$

So we get from Fatou's theorem that $g' \in \mathscr{L}$. We also get that $\int g' \leq \liminf \int g_n \leq B - A$. $\square$

Exercise 19.I. Let U be an open subset of $\mathbb{R}$. Denote by $L(U)$ the sum of the lengths of the components of U. Show that the following are true. If $L(U) < \infty$, then $\xi_U \in \mathscr{L}$ and $\int \xi_U = L(U)$. If $L(U) = \infty$, then $\xi_U \notin \mathscr{L}$.
If $U_1 \subset U_2 \subset \ldots$ are open sets, then $L(\bigcup_n U_n) = \sup_n L(U_n)$.
If $U_1, U_2, \ldots$ are open sets, then $L(\bigcup_n U_n) \leq \sum_n L(U_n)$.

Solution. Notice that the definition of $L(U)$ makes sense: All components of U are open intervals, so we can assign length to any of them. There is only countably many of them. Since $L(U)$ is expressed as a sum of non-negative numbers, the ordering of summands does not influence the value of the sum.

We will use several times the fact that $U \subseteq V$ implies $L(U) \leq L(V)$. (Which is not very difficult to prove.)

a) $L(U)$ *and* $\int \xi_U$. Let I_k, $k \in \mathbb{N}$, be the open intervals which are the components of U. Then we have

$$\sum_{k=1}^{\infty} \xi_{I_k} = \xi_U.$$

If $\boxed{L(I_k) = \infty}$ for some $k \in \mathbb{N}$, then also $L(U) = \infty$ (since $I_k \subseteq U$). In this case we also have $\xi_{I_k} \leq \xi_U$, hence $\xi_U \notin \mathscr{L}$. (Otherwise we would get that $L(I_k) = \int \xi_{I_k} < \infty$ from the monotonicity of integral.)

So in this case the claim is true. From now on we will assume that $L(I_k) < \infty$ for each component I_k of U.

Let us denote

$$s_n = \sum_{k=1}^{n} \xi_{I_k}.$$

We have $s_n \in \mathscr{L}$ and $\int s_n = \sum\limits_{k=1}^{n} L(I_k)$. We also have $\lim\limits_{n\to\infty} s_n = \chi_U$.

If $\boxed{L(U) < \infty}$ then

$$\int s_n \leq L(U)$$

for each n and thus s_n fulfills the assumptions of Monotone convergence theorem (Theorem 19.13). From this theorem we get that $\xi_U \in \mathscr{L}$ and

$$\int \xi_U = \lim_{n\to\infty} \int s_n = \sum_{k=1}^{\infty} L(I_k) = L(U).$$

Now let us assume that $\boxed{L(U) = \infty}$. We want to show that $\xi_U \notin \mathscr{L}$. We will prove this by contradiction.

Assume that $\xi_U \in \mathscr{L}$. Then we have $s_n \leq \xi_U$ for each n and we can apply Lebesgue dominated convergence theorem. From this theorem we get that

$$\int \xi_U = \lim_{n\to\infty} \int s_n = \sum_{k=1}^{\infty} L(I_k) = L(U) = \infty,$$

which contradicts the assumption that $\xi_U \in \mathscr{L}$, i.e., that $\int \xi_U < \infty$.

It is useful to notice that the implication $L(U) = \infty \Rightarrow \xi_U \notin \mathscr{L}$ can be also read as $\xi_U \in \mathscr{L} \Rightarrow L(U) < \infty$. (And so we also have $L(U) = \int \xi_U$ whenever $\xi_U \in \mathscr{L}$.)

b) *Nested sequence of open sets.* Let us denote $U = \bigcup\limits_{n\in\mathbb{N}} U_n$.

If $\boxed{\sup L(U_n) = \infty}$, then also $L(U) = \infty$ (since $L(U_n) \leq L(U)$ for each n). So in this case the inequality holds.

Now let us assume that $\boxed{\sup L(U_n) < \infty}$. We will use monotone convergence theorem for the sequence of function ξ_{U_n}. It is clear that $\lim_{n\to\infty} \xi_{U_n} = \xi_U$. We also get

$$\int \xi_{U_n} = L(U_n) \leq \sup L(U_n).$$

Thus the assumptions of the monotone convergence theorem are fulfilled. So we get that $\xi_U \in \mathscr{L}$ and

$$L(U) = \int \xi_U = \lim_{n\to\infty} \int \xi_{U_n} = \lim_{n\to\infty} L(U_n) = \sup L(U_n).$$

c) *Countable subadditivity.* Let us start by defining

$$V_n := U_n \setminus \left(\bigcup_{k=1}^{n-1} U_k \right) = \left(\bigcup_{k=1}^{n} U_k \right) \setminus \left(\bigcup_{k=1}^{n-1} U_k \right).$$

We clearly have $V_n \subseteq U_n$, the sets V_n are disjoint and

$$\xi_U = \sum \xi_{V_n} \leq \sum \xi_{U_n}.$$

We also have $\xi_{V_n} \in \mathscr{L}$, since $\mathscr{L}$ is closed under finite sums and differences.

If $\sum L(U_n) = \infty$, then the desired inequality holds. So let us assume that $\boxed{\sum L(U_n) < \infty}$.

In this case we get that $\xi_{U_n} \in \mathscr{L}$ and $L(U_n) = \int \xi_{U_n}$. So we get

$$\int \sum_{k=1}^{n} \xi_{V_k} = \sum_{k=1}^{n} \int \xi_{V_k} \leq \sum_{k=1}^{n} \int \xi_{U_k} = \sum_{k=1}^{n} L(U_k) \leq \sum_{k=1}^{\infty} L(U_k).$$

So for the functions $f_n := \sum_{k=1}^{n} \xi_{V_k}$ we get that the sequence $\int f_n$ is bounded and from the monotone convergence theorem we get $\xi_U \in \mathscr{L}$ and

$$L(U) = \int \xi_U = \lim_{n\to\infty} f_n = \sum_{k=1}^{\infty} \xi_{V_k} \leq \sum_{k=1}^{\infty} L(U_k).$$

□

Exercise 19.Q. Let $a \leq b$. Let $\phi\colon [a,b] \to \mathbb{R}$ have a continuous derivative. Assume that $\phi'(x) \geq 0$ for all $x \in [a,b]$. Then

$$\int_{\phi(a)}^{\phi(b)} f(x)\,\mathrm{d}x = \int_a^b f(\phi(x))\phi'(x)\,\mathrm{d}x$$

for every $f \in \mathscr{L}[\phi(a), \phi(b)]$.

Solution. First, let us notice that this equality holds if f is continuous and $\operatorname{supp}(f) \subseteq [\phi(a), \phi(b)]$.

In this case we can use Fundamental theorem of calculus (Theorem 8.1(viii)). We have that there exists an antiderivative F of f and

$$\int_{\phi(a)}^{\phi(b)} f(x)\ \mathrm{d}x = F(\phi(b)) - F(\phi(a)).$$

We also have $F(\phi(x))' = f(\phi(x))\phi'(x)$ and the function $f(\phi(x))\phi'(x)$ is continuous. So using FTC once again we get

$$\int_a^b f(\phi(x))\phi'(x)\ \mathrm{d}x) = F(\phi(b)) - F(\phi(a)).$$

Now if f is Lebesgue integrable over the interval $[\phi(a), \phi(b)]$, then there exist function $\phi_n \in \mathscr{C}_c$ such that $\phi_n \to f \cdot \chi_{[\phi(a),\phi(b)]}$ and $\sum \int |\phi_{n+1} - \phi_n| < +\infty$.

Let us consider a_n, b_n such that $\operatorname{supp}(\phi_n) \subseteq [a_n, b_n]$ for each n.

Let us define

$$\psi_n(x) = \phi_n(\phi(x))\phi'(x).$$

It is clear that $\psi_n(x) \to f(\phi(x))\phi'(x)\chi_{[a,b]}$. Moreover

$$\int |\psi_{n+1} - \psi_n| = \int_{a_n}^{b_n} |\phi_{n+1}(\phi(x))\phi'(x) - \phi_n(\phi(x))\phi'(x)|\ \mathrm{d}x =$$
$$= \int_{a_n}^{b_n} |\phi_{n+1}(\phi(x)) - \phi_n(\phi(x))|\phi'(x)\ \mathrm{d}x = \int_{\phi(a_n)}^{\phi(b_n)} |\phi_{n+1}(x) - \phi_n(x)|\ \mathrm{d}x = \int |\phi_{n+1} - \phi_n|.$$

since we already know that the claim is true for the continuous function $|\phi_{n+1} - \phi_n|$. So we get that $\sum \int |\psi_{n+1} - \psi_n| = \sum \int |\phi_{n+1} - \phi_n| < +\infty$.

Again, using the fact that the claim is true for continuous functions, we get

$$\int \psi_n = \int_{a_n}^{b_n} \phi_n(\phi(x))\phi'(x)\ \mathrm{d}x = \int_{\phi(a_n)}^{\phi(b_n)} \phi_n = \int \phi_n.$$

Hence

$$\int_a^b f(\phi(x))\phi'(x)\ \mathrm{d}x = \lim_{n\to\infty} \int \psi_n = \lim_{n\to\infty} \int \phi_n = \int_{\phi(a)}^{\phi(b)} f(x)\ \mathrm{d}x.$$

□

Exercise 19.S. For $x \in [0, 1]$, let $0.\alpha_1(x)\alpha_2(x)$ be its standard development to the base 2. We want to prove that for 'most' numbers x the sequence $\alpha_1(x), \alpha_2(x), \ldots$ contains 'as many 0s as 1s'.

To this end, we call a number x *regular* if

$$\lim_{n\to\infty} \frac{1}{n} \#\{i; 1 \le i \le n, \alpha_i(x) = 1\} = \frac{1}{2}.$$

Then we claim: *Almost every element of* $[0,1]$ *is regular.*

For the proof we use a slightly different notation. Define functions[39] $f_1, f_2, \dots$ on $[0,1]$ by

$$f_i(x) = \begin{cases} 1 & \text{if } \alpha_i(x) = 1, \\ -1 & \text{if } \alpha_i(x) = 0, \end{cases}$$

and set $F_n := \frac{f_1+\dots+f_n}{n}$. We are done if $\lim\limits_{n\to\infty} F_n = 0$ a.e. One can prove this formula along the following lines.

(i) Show that $\int_0^1 f_i f_j = 0$ if $i \neq j$.

(ii) Deduce from this that $\int_0^1 F_n^2 = 1/n$ for all n.

(iii) Use Theorem 19.11 to show that $\lim\limits_{i\to\infty} F_{i^2} = 0$ a.e. on $[0,1]$.

(iv) Show that, if $n < p \le m$, then $|F_p| \le \frac{m-n}{n} + |F_n|$.

(v) Conclude that $\lim\limits_{n\to\infty} F_n = 0$ a.e. on $[0,1]$.

Without serious problems, similar theorems can be proved for bases other than 2.

Solution. (i) TODO

(ii) TODO

(iii) We have

$$\sum \int |F_{n^2}^2| = \sum \int F_{n^2}^2 = \sum \frac{1}{n^2} < \infty.$$

Thus by Theorem 19.11 the sum $\sum F_{n^2}$ converges a.e., which implies that $\lim\limits_{n\to\infty} F_{n^2} = 0$ a.e.

(iv) $|F_p| = \left|\frac{f_1+\dots+f_p}{p}\right| = \left|\frac{nF_n+f_{p+1}+\dots+f_n}{p}\right| \le \frac{n}{p}\cdot|F_n| + \frac{m-n}{p} \le |F_n| + \frac{m-n}{n}$.

(v) For $n^2 \le k < (n+1)^2$ we get

$$|F_k| \le \frac{(n+1)^2 - n^2}{n^2} + |F_{n^2}|$$

and $\lim\limits_{n\to\infty} \left(\frac{(n+1)^2-n^2}{n^2} + |F_{n^2}|\right) = 0$. □

Exercise 19.U. A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is said to be *separately continuous* if for every $c \in \mathbb{R}$ the functions $x \mapsto f(x,c)$ and $y \mapsto f(c,y)$ are continuous. Prove that every separately continuous function on $\mathbb{R}^2$ is of the first class of Baire. (Show that we may restrict ourselves to a bounded $f\colon \mathbb{R}^2 \to \mathbb{R}$. Define $g\colon \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) := \int_0^y f(x,t)\,\mathrm{d}t$ $(x,y \in \mathbb{R})$. Use dominated convergence to prove that g is continuous and observe that f is a partial derivative of g.)

Sketch of the solution. We have

$$\lim_{n\to\infty} f(x_n,t)\chi_{[0,y_n]}(t) = f(x,t)\chi_{[0,y]}(t)$$

[39] IIRC this system of functions is called *Haar system.* In [KK, Exercise 4.2.4], [ŠŠN, Section 12.3] it is called *Rademacher system.*

almost everywhere. (The only possible exception is the point y.)

Therefore

$$\lim_{n\to\infty}\int_0^{y_n} f(x_n,t)\,\mathrm{d}t = \int_0^t f(x,t)\,\mathrm{d}t.$$

Moreover we have

$$\frac{g(x,y+\delta)-g(x,y)}{\delta} = \frac{\int_y^{y+\delta} f(x,t)\,\mathrm{d}t}{\delta} = f(x,t_\delta)$$

for some t_δ between y and $y+\delta$.

Taking the limit $\delta\to 0$ and using separate continuity we get

$$\frac{\partial g(x,y)}{\partial y} = f(x,y).$$

We know that the derivative belongs to the first class of Baire; similar proof works for partial derivative. □

Exercise 19.V. (The indefinite lower Riemann integral.) Let $f\colon [a,b]\to\mathbb{R}$ be bounded. If f is Riemann integrable, then oneeasily sees (e.g. Exercise 12.O or Theorem 15.6) that the indefinite integral Jf of f is increasing if and only if $f\ge 0$ a.e. If f is not Riemann integrable we can still talk about its 'indefinite lower Riemann integral', the function $J_-f\colon x\mapsto \int a^x f(t)\,\mathrm{d}t$. Even if $f\ge 0$ a.e., this J_-f may fail to be increasing. (Consider $-\xi_{\mathbb{Q}}$.) In this exercise we investigate J_-f for any bounded function f. The Lebesgue integral will be a useful tool.

Let $f\colon [a,b]\to\mathbb{R}$ be bounded. We use the symbols $f^\uparrow$ and $f^\downarrow$ as in Definition 10.4 and ω as in Exercise 10.K. Observe that $f^\uparrow$, $f^\downarrow$, ω are Lebesgue integrable over $[a,b]$ (Corollary 19.16).

For $x\in[a,b]$, $J_+f(x)$ and $J_-f(x)$ will denote the upper and lower Riemann integrals of f over $[a,x]$, respectively.

(i) Show that

$$J_+f(b) = \int_a^b f^\downarrow, \qquad J_-f(b) = \int_a^b f^\uparrow, \qquad J_+f(b) - J_f(b) = \int_a^b \omega$$

(For the first formula, note that there exist continuous functions $g_1, g_2, \dots$ with $g_1\ge g_2\ge\dots$ and $\lim_{n\to\infty} g_n = f^\downarrow$. Apply the monotone convergence theorem.)

(ii) Give a new proof of Theorem 12.1.

(iii) Show that J_-f is increasing if and only if for every $\varepsilon>0$ the closure of $\{x; f(x)\le -\varepsilon\}$ is a nullset.

(iv) Set $f^-(x) := \max\{-f(x),0\}$ ($x\in[a,b]$). Show that J_-f is increasing if and only if f^- is Riemann integrable over $[a,b]$ with integral 0.

Solution. Some things we know about $f^\uparrow$, $f^\downarrow$, ω:

- $f^\uparrow = \sup\{g(x); g\le f; g\in\mathcal{C}^+\}$
- $f^\uparrow(t) = \lim_{n\to\infty}\inf\{f(x) : x\in I, |x-t|\le 1/n\}$

- $f^\uparrow \le f \le f^\downarrow$
- $f^\uparrow \in \mathcal{C}^+$ and thus by Theorem 10.6 there is an increasing sequence of continuous functions converging to f pointwise.
- $\omega = f^\uparrow - f^\downarrow \in \mathcal{C}^-$
- $\omega(x) = 0$ if and only if f is continuous at x.

(i) Since $f^\downarrow$ is pointwise limit of continuous functions, the integral $\int_a^b f^\downarrow$ exists.

Suppose we have a partition $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$ and numbers h_i such that $f(x) \le h_i$ for $x \in (x_{i-1}, x_i)$. Then for each $x \in (x_{i-1}, x_i)$ we also have $f^\downarrow(x) \le h_i$, which means that

$$\int f^\downarrow \le \sum_{i=1}^{n} h_i(x_i - x_{i-1})$$

and thus

$$\int_a^b f^\downarrow \le \overline{\int_a^b} f.$$

On the other hand, we also have $f \le f^\downarrow$, hence

$$\overline{\int_a^b} f \le \overline{\int_a^b} f^\downarrow = \int_a^b f^\downarrow.$$

(ii) Theorem 12.1: For a bounded function $f\colon [a, b] \to \mathbb{R}$ we have: The function f is Riemann integrable $\Leftrightarrow$ the set of points of discontinuity of f is a nullset ($=$ f is continuous almost everywhere.)

The function f is Riemann integrable if and only

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

which is equivalent to $\int_a^b f^\downarrow = \int_a^b f^\uparrow$, i.e.,

$$\int_a^b f^\downarrow - f^\uparrow = \int_a^b \omega = 0.$$

By Corollary 19.12 this means that $\omega = 0$ a.e., i.e., that f is continuous a.e.

(iii) First, notice that $\{x; f^\uparrow(x) < -\varepsilon\} \subseteq \overline{\{x; f(x) \le -\varepsilon\}} \subseteq \{x; f^\uparrow(x) \le -\varepsilon\}$.

Thus the assumption is equivalent to saying that, for each $\varepsilon > 0$, we have $f^\uparrow(x) + \varepsilon \ge 0$ a.e.

Now, according to Corollary 19.12, this is equivalent to

$$Jf(b) - Jf(a) + \varepsilon(b - a) = \int_a^b (f^\uparrow + \varepsilon) \ge 0,$$

which means that $Jf(b) - Jf(a) \ge -\varepsilon(b - a)$.

This is true for every $\varepsilon > 0$ if and only if $Jf(b) - Jf(a) \geq 0$, i.e., $Jf(b) \geq Jf(a)$.

(iv) By Exercise 4.C we know that $\overline{\{}\{x; f(x) \leq -\varepsilon\}\}$ is a null set if and only if $\{x; f(x) \leq -\varepsilon\}$ has zero content.

So if $J_- f$ is increasing then $\{x; f(x) \leq -\varepsilon\}$ can be covered by finitely many intervals of total length ε_1 (for any given ε_1). From these intervals we can get a partition such that the overestimate for the upper Riemman sum of f^- is

$$\varepsilon(b-a) + M\varepsilon_1$$

(where $M = \sup\{|f(x)|; x \in [a,b]\}$). Since ε and ε_1 can be chosen arbitrarily small, we get that f^- is Riemann integrable and $\int f^- = 0$.

Conversely, if f^- is Riemann integrable and the value of Riemann integral is 0, then for any given ε_1 there is a partition such that the upper Riemann sum is at most ε_1. If we consider only the intervals on which a value greater than ε is attained, we get the upper Riemann sum is at least $L\varepsilon$, where L denotes the total length of these interval. So we have inequality $\varepsilon_1 \geq L\varepsilon$, i.e.,

$$L \geq \frac{\varepsilon_1}{\varepsilon},$$

and we know that $\{x; f(x) \leq -\varepsilon\}$ can be covered by finitely many intervals of total length L.

Since ε_1 and ε are arbitrary positive numbers, we can choose them to get L arbitrarily small. This means that $\{x; f(x) \leq -\varepsilon\}$ has zero content, $\overline{\{x; f(x) \leq -\varepsilon\}}$ is a null set and by (iii) $J_- f$ is increasing. $\square$

6.20 Lebesgue measurability

On choice of ϕ_n in the proof of Lemma 20.1. Directly from Lemma 19.10 we have that for $\varepsilon = 4^{-n}$ there exists $\phi_n \in \mathscr{C}_c$ such that $\int |f - \phi_n| \leq 4^{-n}$. Then we have

$$\begin{aligned}\int |\phi_n - \phi_{n+1}| &= \int |(f - \phi_n) - (f - \phi_{n+1})| \\ &\leq \int |f - \phi_n| + \int |f - \phi_{n+1}| \leq \frac{3}{4^{n+1}}.\end{aligned}$$

Thus $\sum \int |\phi_n - \phi_{n+1}|$ converges and from Theorem 19.11 we get that ϕ_n converges to some function g and also

$$\lim_{n\to\infty} \int \phi_n = \int g.$$

For this function have

$$\int |f - g| \leq \int |f - \phi_n| + \int |g - \phi_n| < \varepsilon,$$

where n is chosen in a such way that $\int |f - \phi_n| < \varepsilon/2$ and $\int |g - \phi_n| < \varepsilon/2$.

Since this is true for every positive ε, we get

$$\int |f-g| = 0.$$

By Corollary 19.12 this implies that $f = g$ a.e.

Theorem 20.9. We have $A - \{x \in C; \chi_E(x) = 1\} = E \cap C$. This set is closed and it is also relatively open. Thus there exists an openset such that $A = W_1 \cap C$. We put $W = W_1 \cup (\mathbb{R} \setminus C)$.

Inclusion $A \subseteq E \subseteq W$. The inclusion $A \subseteq E$ follows from $A = E \cap C$. We also have

$$E = (E \cap C) \cup (E \setminus C) = A \cup (E \setminus C) \subseteq W_1 \cup (\mathbb{R} \setminus C) = W.$$

Inequality $\lambda(W \setminus A) < \varepsilon$. We have

$$W \setminus A = (W \setminus W_1) \cup (W \setminus C).$$

Now $W \setminus W_1 = (W_1 \cup (\mathbb{R} \setminus C)) \setminus W_1 \subseteq \mathbb{R} \setminus C$. And also $W \setminus C \subseteq \mathbb{R} \setminus C$. Together we get

$$W \setminus A = (W \setminus W_1) \cup (W \setminus C) \subseteq \mathbb{R} \setminus C$$

and $\lambda(W \setminus A) \leq \lambda(\mathbb{R} \setminus C) < \varepsilon$.

Exercise 20.O. (Indefinite integrals) Let I be an interval, $f \in \mathscr{L}(I)$. A function $F\colon I \to \mathbb{R}$ is called an *indefinite integral* of f if

$$F(y) - F(x) = \int_x^y f(t)\,\mathrm{d}t \qquad (x, y \in I, x < y)$$

(i) Show that f has an indefinite integral, that any two indefinite integrals of f differ by a constant, and that all indefinite integrals of f are continuous. (Prove that $\lim\limits_{n\to\infty} F(y_n) = F(y)$ if $\lim\limits_{n\to\infty} y_n = y$.)
TODO (ii), (iii), (iv)

Exercise 20.P. Not every measurable function on I. is locally integrable. More than that: there exists a positive Lebesgue measurable function on $\mathbb{R}$ that is integrable over no interval. (Hint for the proof of the last statement. Let $(r_1, r_2, \dots)$ be an enumeration of $\mathbb{Q}$. Define $g \in \mathscr{L}$ by $g(x) := x^{-1/2}$ if $0 < x \leq 1$, $g(x) := 0$ if $x = 0$ or $|x| > 1$. Then $\sum\limits_{n=1}^{\infty} 2^{-n} g(x - r_n)$ is finite for almost every $x \in \mathbb{R}$. The a.e. defined function $x \mapsto \left(\sum\limits_{n=1}^{\infty} 2^{-n} g(x - r_n)\right)^2$ is integrable over no interval.)
(ii) Extend the previous exercise to locally integrable functions.

Solution. (ii) Existence of nowhere locally integrable function.[40] We have

$$g(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \in (0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\int_0^1 g(x)\,\mathrm{d}x = 2$.

The sum

$$\sum_{n=1}^{\infty} 2^{-n} g(x - r_n)$$

is a.e. finite by Theorem 19.11, since $\sum\limits_{n=1}^{\infty} 2^{-n} \int g(x - r_n)\,\mathrm{d}x < +\infty$.

However, every interval contains interval of the form $(r_n - \varepsilon, r_n + \varepsilon)$ for some rational number r_n and some $\varepsilon > 0$ and

$$\int_{r-\varepsilon}^{r+\varepsilon} g^2(x - r_n)\,\mathrm{d}x = \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{d}x}{x} = +\infty.$$

□

6.21 Absolute continuity

Exercise 21.J. (i) If $f\colon [a,b] \to [A,B]$ is absolutely continuous and if $\phi\colon [A,B] \to \mathbb{R}$ satisfies a Lipschitz condition, then $\phi \circ f$ is absolutely continuous (see, however, Exercise 21.L). If f and g are absolutely continuous, then so are $|f|$, $f \vee g$ and $f \wedge g$.
(ii) The product of two absolutely continuous functions is absolutely continuous.

Exercise 21.K. If $f \in \mathscr{BV}[a,b]$, then then the almost everywhere defined function f' is Lebesgue integrable and $\int_a^b |f'| \le \operatorname{Var} f$. In particular, if f is increasing, then $\int_a^b f' \le f(b) - f(a)$. (Hint. If f is increasing, adapt Exercise 19.H to show that $\int_a^b f' \le f(b) - f(a)$. For the general case, consider the indefinite variation T of f (Exercise 3.B) and note that $|f'| \le T$ a.e.)

Exercise 21.L. In Exercise 3.F we obtained a differentiable function $g\colon [0,1] \to [0,1]$ that is not of bounded variation while g^2 is. Deduce the existence of absolutely continuous functions $h, j\colon [0,1] \to [0,1]$ for which $h \circ j$ is not absolutely continuous and not even of bounded variation.

Exercise 21.N. Let $f\colon [a,b] \to \mathbb{R}$ be continuous. In Exercise 15.H we saw that f is increasing (strictly increasing) as soon as $D^- f(x) \ge 0$ ($D^- f(x) > 0$) for every $x \in [a,b]$. We have not yet answered the natural question: if $D^- f = 0$, must f be constant? We are now in a position to do so.

[40] `http://math.stackexchange.com/q/24413`: Is there a function with infinite integral on every interval?

(i) Let H, K be differentiable functions on $[a,b]$ such that $H' \leq D^- f \leq K'$. Then

$$H(y) - H(x) \leq f(y) - f(x) \leq K(y) - K(x) \qquad (a \leq x \leq y \leq b) \qquad (*)$$

We give an outline of a proof, leaving the details to the reader. Without loss of generality, assume $K' = 0$. Then H is decreasing. As $D^-(f-H) = D^- f - H \geq 0$, $f - H$ is increasing (Exercise 15.F). This proves half of $(*)$. (We need this half to prove the rest, i.e. that f is decreasing.) The function $g\colon x \mapsto (f-2H)(x)+x$ is a strictly increasing map of $[a,b]$ onto a closed interval $[A,B]$. Let ϕ be its inverse map and set $f_1 := f \circ \phi$. It suffices to prove that f_1 is decreasing. Now ϕ is strictly increasing and

$$\phi(v) - \phi(u) \leq v - u \qquad (A \leq u \leq v \leq B).$$

From the fact that $D^- f \leq 0$ it follows easily that $D^- f_1 = D^-(f \circ \phi) \leq 0$.

Now note that for all $x \in [a,b]$ we have $(g-f)(x) = -2H(x) + x$ and $(g+f)(x) = 2(f-H)(x) + x$, so that both $g - f$ and $g + f$ are increasing. Therefore, if $a \leq x \leq y \leq b$, then $|f(y) - f(x)| \leq g(y) - g(x)$. Consequently, f_1 satisfies a Lipschitz condition. In particular, f_1 is absolutely continuous, hence is an indefinite integral of f_1'. But $f_1' = D^- f$ for every u for which $f'(u)$ exists. Thus, $f_1' \leq 0$ a.e. and f_1 is decreasing,
(ii) In particular, if $D^- f = 0$, then f is constant.
(iii) In Exercise 15.H we have already observed that, even if $D^- f(x) \leq$ for every x, f may still not be decreasing. Paradoxically, in (i) we have proved: if there exists a differentiable function H such that $H'(x) \leq D^- f(x) \leq 0$ $(a \leq x \leq b)$, then f is decreasing.

Solution. Recall that

$$D^- f(x) = \liminf_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

(i) TODO W.l.o.g. $K' = 0$. TODO
This means that we assume that $H' \leq D^- f \leq 0$ and we want to prove

$$H(y) - H(x) \leq f(y) - f(x) \leq 0 \qquad (a \leq x \leq y \leq b) \qquad (**)$$

Since both f and H are continuous, so is the function $g(x) = (f-2H)(x)+x$. We also know that both $(-H)$ and $(f-H)$ are increasing. Notice that

$$g = (f - H) + (-H) + id.$$

So we have written g as a sum of increasing and strictly increasing function, which means that g is strictly increasing.

From the continuity and monotonicity of g we get that it is a bijection from $[a,b]$ to $[A,B]$ and we can define $\phi = g^{-1}$, which is also strictly increasing.

We also know that $g(x) - x = (f - 2H)(x)$ in increasing, meaning that $g(x) - x \le g(y) - y$, meaning that

$$y - x \le g(y) - g(x).$$

If we put $y = \phi(v)$ and $x = \phi(u)$, we get that

$$\phi(v) - \phi(u) \le v - u.$$

Now we use the fact that the following functions are strictly increasing:

$$\begin{aligned} g - f &= (-2H) + id \\ g + f &= 2(f - H) + id \end{aligned}$$

(Each of them is written as a sum of an increasing and a strictly increasing function.) So we get

$$\begin{aligned} g(x) - f(x) &\le g(y) - f(y) \\ g(x) + f(x) &\le g(y) + f(y) \end{aligned}$$

which is equivalent to

$$\begin{aligned} f(y) - f(x) &\le g(y) - g(x) \\ g(x) - g(y) &\le f(y) - f(x) \end{aligned}$$

So we got

$$-(g(y) - g(x)) \le f(y) - f(x) \le g(y) - g(x),$$

which is the same as

$$|f(y) - f(x)| \le g(y) - g(x).$$

Now it only remains to notice that

$$|f_1(v) - f_1(u)| = |f(\phi(v)) - f(\phi(u))| \le |g(\phi(v)) - g(\phi(u))| = |v - u|.$$

TODO □

Exercise 21.O. (Generalization of 14.I.) Let $g\colon [0,1] \to \mathbb{R}$ be absolutely continuous. Then $fg \in \mathscr{D}'[0,1]$ for every $f \in \mathscr{D}'[0,1]$. In fact, if $f \in \mathscr{D}'[0,1]$ and if F is an antiderivative of f, then

$$h\colon x \mapsto F(x)g(x) - \int_0^x F(t)g'(t)\,\mathrm{d}t$$

is an antiderivative of fg. Hint. If $0 \le x \le y \le 1$, then

$$\frac{h(y) - h(x)}{y - x} = \frac{F(y) - F(x)}{y - x} g(y) - \int_x^y \frac{F(t) - F(x)}{t - x} \cdot \frac{t - x}{y - x} \cdot g'(t)\,\mathrm{d}t$$

See also Exercise 14.F(ii).

Solution. Since g has bounded variation, it is differentiable a.e. on $[0,1]$ (Theorem 21.21(ii)), so we can work with the function $g'\colon [0,1] \to \mathbb{R}$. (In the points where g is not differentiable, we choose zero value. This does not influence the integrals below.)

For $h(x) = F(x)g(x) - \int_0^x F(t)g'(t)\,\mathrm{d}t$ we have

$$\begin{aligned}
h(y) - h(x) &= F(y)g(y) - F(x)g(x) - \int_x^y F(t)g'(t)\,\mathrm{d}t \\
&= (F(y) - F(x))g(y) + F(x)(g(y) - g(x)) - \int_x^y F(t)g'(t)\,\mathrm{d}t \\
&= (F(y) - F(x))g(y) + F(x)\int_x^y g'(t)\,\mathrm{d}t - \int_x^y F(t)g'(t)\,\mathrm{d}t \\
&= (F(y) - F(x))g(y) - \int_x^y (F(t) - F(x))g'(t)\,\mathrm{d}t
\end{aligned}$$

which gives us

$$\frac{h(y) - h(x)}{y - x} = \frac{F(y) - F(x)}{y - x}g(y) - \int_x^y \frac{F(t) - F(x)}{t - x} \cdot \frac{t - x}{y - x}g'(t)\,\mathrm{d}t$$

We know that

$$\lim_{y \to x} \frac{F(y) - F(x)}{y - x} = f(x).$$

So for any $\varepsilon > 0$ we may choose $\delta > 0$ such that $|y - x| < \delta$ implies $\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| \le \varepsilon$ and, consequently,

$$\left|\frac{F(y) - F(x)}{y - x}\right| \le |f(x)| + \varepsilon.$$

For t between x and y we also have

$$0 \le \frac{t - x}{y - x} \le 1$$

From this we get the estimate

$$\left|\int_x^y \frac{F(t) - F(x)}{t - x} \cdot \frac{t - x}{y - x}g'(t)\,\mathrm{d}t\right| \le \int_x^y \left|\frac{F(t) - F(x)}{t - x}\right| \cdot \frac{t - x}{y - x}|g'(t)|\,\mathrm{d}t \le (|f(x)| + \varepsilon)\int_x^y |g'(t)|\,\mathrm{d}t.$$

By Lemma 21.17 the function $y \mapsto \int_x^y |g'(t)|\,\mathrm{d}t$ is absolutely continuous, hence it is also continuous. This implies that for $y \to x$ the above expression tends to 0.

Together we get that

$$h'(x) = \lim_{y \to x} \frac{h(y) - h(x)}{y - x} = \lim_{y \to x} \frac{F(y) - F(x)}{y - x}g(y) = f(x)g(x).$$

□

Exercise 21.P. (The arc length, again) For a continuous function $f\colon [a,b] \to \mathbb{R}$ we denote $L(f)$ the length of its graph (see Section 3). We already know (Definition 3.1 and Exercise 3.E):
(i) $L(f)$ is finite if and only if f is of bounded variation.
(ii) If f has a continuous derivative, then $L(f) = \int_a^b \sqrt{1+(f')^2}$.
We now prove:
(iii) If f is of bounded variation, then $L(f) \geq \int_a^b \sqrt{1+(f')^2}$.
(iv) If f is absolutely continuous, then $L(f) = \int_a^b \sqrt{1+(f')^2}$.
To prove (iii): For $x \in [a,b]$ let $l(x)$ be the length of the graph of f, restricted to $[a,x]$. Show that l is increasing and that

$$l(y) - l(x) \geq \sqrt{(y-x)^2 + (f(y)-f(x)^2)}, \qquad a \leq x \leq y \leq b.$$

Deduce that $l' \geq \sqrt{1+(f')^2}$ and apply Exercise 21.K.
To prove (iv): Let $\varepsilon > 0$. Choose a partition P of $[a,b]$ with $L_P(f) \geq L(f) - \varepsilon$. (For $L_P(f)$, see Section 3.) Choose a continuous function ϕ on $[a,b]$ such that $\int_a^b |f' - \phi|$ is sufficiently small (Lemma 19.10) and show that $|L_P(f) - L_P(\Phi)|$ is very small, where Φ is an indefinite integral of ϕ. Prove that $|\int_a^b \sqrt{1+(f')^2} - \int_a^b \sqrt{1+\phi^2}| \leq \int_a^b |f' - \phi|$ and observe that $L_P(\Phi) \leq L(\Phi) = \int_a^b \sqrt{1+\phi^2}$.

Exercise 21.Q. Compute the length of the graph of Cantor function.

Lemma. *Let $f\colon [a,b] \to \mathbb{R}$ be increasing. Then*

$$L(f) = (b-a) + (f(b) - f(a)).$$

Proof. It suffices to notice that for any partition P we have $L_p \leq |b-a| + |f(b) - f(a)|$. Indeed

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \leq \sum_{i=1}^{n} (|x_i - x_{i-1}| + |f(x_i) - f(x_{i-1})|) =$$
$$= \sum_{i=1}^{n} (x_i - x_{i-1}) + \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = (b-a) + (f(b) - f(a)).$$

□

Solution. TODO $L(f) = 2$ □

Exercise 21.R Let $g, f_1, f_2, \ldots$ be continuous functions on $[a,b]$ such that $g = \lim_{n\to\infty} f_n$. Show that

$$L(g) \leq \liminf_{n\to\infty} L(f_n).$$

Solution. Let Δ denotes the set of all partitions of the interval $[a,b]$.

Directly from the definition of $L_P(f)$ we get that $\lim_{n\to\infty} L_P(f_n) = L_P(g)$ holds for any partition $P \in \Delta$.

(It suffices to notice that the function

$$(y_0, \dots, y_k) \mapsto \sum_{i=1}^{k} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

from $\mathbb{R}^{k+1}$ to $\mathbb{R}$ is continuous and $L_P(f)$ is obtained by plugging $(f(x_0), \dots, f(x_k))$ into this function.)

By definition,

$$L(g) = \sup_{P \in \Delta} L_P(g)$$

which implies that

$$(\forall \varepsilon > 0)(\exists P \in \Delta) L(g) - \varepsilon < L_P(g)$$

Now if we use that $\lim_{n \to \infty} L_P(f_n) = L_P(g)$, then we get

$$(\forall \varepsilon > 0)(\exists P \in \Delta)(\exists n_0)(\forall n \geq n_0) L(g) - \varepsilon < L_P(f_n).$$

But we also have $L_P(f_n) \leq L(f_n)$ for every n, which implies

$$(\forall \varepsilon > 0)(\exists P \in \Delta)(\exists n_0)(\forall n \geq n_0) L(g) - \varepsilon < L(f_n).$$

But since neither of the expressions $L(g) - \varepsilon$ and $L(f_n)$ depend on P we can simply write:

$$\begin{aligned}
(\forall \varepsilon > 0)(\exists n_0)(\forall n \geq n_0) L(g) - \varepsilon &< L(f_n) \\
(\forall \varepsilon > 0)(\exists n_0) L(g) - \varepsilon &\leq \inf_{n \geq n_0} L(f_n) \\
(\forall \varepsilon > 0) L(g) - \varepsilon &\leq \sup_{n_0 \in \mathbb{N}} \inf_{n \geq n_0} L(f_n) \\
(\forall \varepsilon > 0) L(g) - \varepsilon &\leq \liminf_{n \to \infty} L(f_n) \\
L(g) &\leq \liminf_{n \to \infty} L(f_n)
\end{aligned}$$

[41] $\square$

Notice that using similar arguments we can show for any P the function $f \mapsto L_P(f)$ is continuous. (Considered as the function from the space $\mathbb{R}^{[a,b]}$ of all functions from $[a, b]$ to $\mathbb{R}$. On this space we take the topology of pointwise convergence, i.e., the product topology.) And now we can use the fact that supremum of lower semicontinuous function is lower semicontinuous. (However, in this book this fact was only shown for functions defined on an interval.) If we know that this is true for arbitrary topological space, in this case we get that $L = \sup_{P \in \Delta}$ is lower semicontinuous (w.r.t. the topology of pointwise convergence.).

[41] My questions:
I do not see where the continuity of these functions is used.
How does the proof work in the case $L(g) = \infty$?

The inequality proved in Exercises 21.R can be strict. For example we can take the function $f(x) = |1 - x|$ on the interval $[0, 2]$. Then we can extend this function to a periodic function with period 2.

If we define $f_n(x) = \frac{1}{2^n} f(2^n x)$, then we get $f_n(x) \to 0$. We also have $L(f_n) = 2\sqrt{2}$, while the limit function $g(x) = 0$ has the graph of length $L(g) = 0$.

Exercise 21.S. For $f\colon \mathbb{R} \to \mathbb{R}$ and $s \in \mathbb{R}$ define $f_s\colon \mathbb{R} \to \mathbb{R}$ by

$$f_s(x) := f(x - s) \qquad (x \in \mathbb{R}).$$

(i) Let f be a Lebesgue measurable function such that for all $s \in \mathbb{R}$ one has $f = f_s$ a.e. Then there exists a number c such that $f = c$ a.e. (For the proof, one may assume f to be bounded. Let F be the indefinite integral of f with $F(0) = 0$. Set $c := F(1)$. From the given property of f it follows that $F(b) - F(a) = F(s + b) - F(s + a)$ $(a, b, s \in \mathbb{R};\ a < b)$ Deduce that $F(s + t) = F(s) + F(t)$ for all $s, t \in \mathbb{R}$. Prove that $F(s) = sF(1)$ for all $s \in \mathbb{Q}$ and, by continuity, for all $s \in \mathbb{R}$. It follows that F is an indefinite integral of the function $x \mapsto c$. Apply Exercise 20.O(i).)
(ii) (A strengthening of (i)) Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and let $S := \{s \in \mathbb{R}; f = f_s \text{ a.e.}\}$ be dense in $\mathbb{R}$. Then there is a $c \in \mathbb{R}$ with $f = c$ a.e. (Proceeding as above one obtains $F(b) - F(a) = F(s + b) - F(s + a)$ $(a, b \in \mathbb{R};$ $a < b)$ for $s \in S$. Then, since F is continuous, the same identity holds for all $c \in \mathbb{R}$.)[42]

Solution. If s is bounded. Recall that from 20.O(i) we know that F (and any indefinite integral of f) is continuous and

$$F(b) - F(a) = \int_a^b f(t)\,\mathrm{d}t$$

(i) We have

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,\mathrm{d}t = \int_{a+s}^{b+s} f_s(t)\,\mathrm{d}t = \\ &= \int_{a+s}^{b+s} f(t)\,\mathrm{d}t = F(b + s) - F(a + s) \end{aligned}$$

Now if we put $a = 0$ and $b = t$, then we get $F(t) = F(s + t) - F(s)$, i.e.,

$$F(s + t) = F(s) + F(t).$$

From this we can easily show that $F(s) = sF(1) = cs$ for $c \in \mathbb{Q}$. Since F is continuous, we get that $F(s) = sF(1) = cs$ for any real number s.

We see that F is an indefinite integral of the constant function c. And from 20.O(iv) we get that $f = c$ almost everywhere.

[42] Probably a typo: "$c \in \mathbb{R}$" should be "$s \in \mathbb{R}$".

(ii) We have

$$F(b) - F(a) = F(s+b) - F(s+a)$$

for each $s \in S$ (using the same argument as above).

The RHS is continuous in s and S is a dense set, therefore the above is true for every $s \in \mathbb{R}$. Now we can proceed exactly in the same way as in (i).

If f is unboudned. TODO $\square$

Exercise 21.T Let $g\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and such that

$$g(x+y) = g(x) + g(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

Then there exists a $c \in \mathbb{R}$ such that $g(x) = cx$ for all $x \in \mathbb{R}$. (Take $c := g(1)$. First show that $g(x) = cx$ for all $x \in \mathbb{N}$, all $x \in \mathbb{Z}$ and all $x \in \mathbb{Q}$. Set $f(x) := g(x) - cx$ and apply the preceding exercise.)

Solution. It is relatively easy to see:
$g(0) = g(x) + g(-x) \Rightarrow g(-x) = -g(x)$
By induction we can show that $g(nx) = ng(x)$ for $n \in \mathbb{N}$. Using $g(-x) = -g(x)$ we get that the same is true for $n \in \mathbb{Z}$.
This also implies $g(x) = g\left(n \cdot \frac{x}{n}\right) = n \cdot g\left(\frac{x}{n}\right)$, which implies $g\left(\frac{x}{n}\right) = \frac{g(x)}{n}$.
Combining the above we get

$$g\left(\frac{p}{q}x\right) = \frac{p}{q}g(x)$$

for any $\frac{p}{q} \in \mathbb{Q}$ and $x \in \mathbb{R}$. In particular, for $x = 1$ and $c = g(1)$ we obtain that

$$g\left(\frac{p}{q}\right) = c \cdot \frac{p}{q},$$

hence $g(x) = cx$ holds for any $x \in \mathbb{Q}$.

Let $f(x) = g(x) - cx$. Notice that the function $f(x)$ fulfills the same functional equation

$$(\forall x, y \in \mathbb{R}) f(x+y) = f(x) + f(y).$$

We have $f(x) = 0$ for each $x \in \mathbb{Q}$ and also

$$f(x+q) = f(x) + f(q) = f(x)$$

which means that $f_q = f$ for every $q \in \mathbb{Q}$.

Since $\mathbb{Q}$ is dense, exercise 21.S implies that there is some constant d such that $f(x) = d$ a.e. In the other words, we have

$$(\forall x \in M) f(x) = d,$$

where $M = \mathbb{R} \setminus A$ and A is a nullset.

If we choose any $x \in \mathbb{R}$, we get

$$M \cap (x + M) \neq \emptyset.$$

(If not, then we would get $\mathbb{R} = \mathbb{R} \setminus (M \cap (x+M)) = (\mathbb{R} \setminus M) \cup (\mathbb{R} \setminus (x+M))$, i.e., $\mathbb{R}$ would be a union of two nullsets.)

This implies that there is a $y \in M$ such that $x + y \in M$. For such x and y we get

$$\begin{aligned} f(x) + f(y) &= f(x+y) \\ f(x) + d &= d \\ f(x) &= 0 \end{aligned}$$

We have shown that $f(x) = g(x) - cx = 0$ for each $x \in \mathbb{R}$, which means that

$$g(x) = cx.$$

□

Different proof, that measurable solution of Cauchy functional equation is continuous can be found in [H, Theorem 5.5].[43]

Exercise 21.U *Let G be a Lebesgue measurable subset of $\mathbb{R}$ that is a group under addition. Then either $G = \mathbb{R}$ or G is a null set.* (If $G \cap (0, \infty)$ has a smallest element, a, say, then $G = \{na; n \in \mathbb{Z}\}$, so G is countable. Otherwise, G is dense.[44] Apply Exercise 21.S to show that either G or $\mathbb{R} \setminus G$ is a null set. But if $\mathbb{R} \setminus G$ is null, then for every $x \in \mathbb{R}$ the sets G and $x + G$ have nonempty intersection, so $x \in G - G = G$.)[45]

Solution. If G does not have the smallest positive element, then there exists $x \in (0, \varepsilon) \cap G$ for any $\varepsilon > 0$. Now if $I \subseteq \mathbb{R}$ is any interval and the length of interval is at least ε, then it contains some nx, where $n \in \mathbb{Z}$, for $x \in G$ chosen in such way that $0 < x < \varepsilon$.

This implies that for the indicator function χ_G we have

$$\chi_G(x) = \chi_G(x - s)$$

for every $s \in G$.

Since G is a dense set, according to Exercise 21.S the function $\chi_G(x)$ is constant a.e.

If $\chi_G(x) = 1$ a.e., then $\mathbb{R} \setminus G$ is a nullset. If $\chi_G(x) = 0$ a.e., then G is a nullset. (These are the only two possibilities, since for $c \neq 0, 1$ we have $\{x \in \mathbb{R}; \chi_G(x) \neq c\} = \mathbb{R}$, which is not a nullset.)

If $\mathbb{R} \setminus G$ is a nullset, then so is its translation $\mathbb{R} \setminus (x + G)$ (for any $x \in \mathbb{R}$). Hence their union $(\mathbb{R} \setminus G) \cup (\mathbb{R} \setminus (x + G)) = \mathbb{R} \setminus (G \cap x + G)$ is also a nullset. This implies that $G \cap x + G \neq \emptyset$, which means $x \in G - G = G$. □

[43] See also: Measurable Cauchy Function is Continuous `http://math.stackexchange.com/q/359183`, Show that $f(x+y) = f(x) + f(y)$ implies f continuous $\Leftrightarrow$ f measurable `http://math.stackexchange.com/q/45861`

[44] Subgroup of $\mathbb{R}$ is either dense or has a least positive element: `http://math.stackexchange.com/q/90177`

[45] Proper measurable subgroups of $\mathbb{R}$: `http://math.stackexchange.com/q/1499`

This result can be also derived from Steinhaus theorem. A proof of Steinhaus theorem can be found, for example, in [AE, Theorem 5.18], [Ku, Section 3.7], [O, Theorem 4.8].[46]

Theorem (Steinhaus)**.** *Let $A \subseteq \mathbb{R}$ is a Lebesgue measurable set of positive measure. Then the set*

$$A - A = \{x - y; x, y \in A\}$$

is a neighborhood of zero, i.e., there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq G - G$.

If we apply this theorem to a group G, then we get that $G = G - G \supseteq (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Using the fact that G is a group we get that $G \supseteq (-n\varepsilon, n\varepsilon)$ for every $n \in \mathbb{N}$. This implies that $G = \mathbb{R}$.

Exercise 21.V We call $f\colon \mathbb{R} \to \mathbb{R}$ *doubly periodic* if there exist $s, t \in \mathbb{R}$ with $f = f_s = f_t$ (see Exercise 21.S) and there does not exist a $p \in \mathbb{R}$ such that both s and t are multiples of p. Prove: if f is doubly periodic and Lebesgue measurable, then there exists a $c \in \mathbb{R}$ with $f = c$ a.e.

Solution. Let us denote

$$G = \{s \in \mathbb{R}; f_s = f\}.$$

It is relatively easy to show that G is a subgroup of $\mathbb{R}$.

This subgroups does not have the smallest positive element. (If $p = \min G \cap (0, \infty)$, then we get that both s and t are multiples of p.)

So from the first part of Exercise 21.U we get that G is dense. Since we have

$$(\forall x \in G) f_s = f,$$

Exercise 21.S(ii) implies that $f = c$ a.e (for some $c \in \mathbb{R}$). □

6.22 The Perron integral

Exercise 22.A. (iii) Let $f\colon [a, b] \to \mathbb{R}$ be Lebesgue integrable and upper semicontinuous. If v is an indefinite integral of f, then $D^+ v \leq f$.

Solution. Since v is indefinite integral of f, we have

$$v(y) - v(x) = \int_x^y f(t)\, \mathrm{d}t$$

for any $x, y \in [a, b]$.

[46] See also `http://math.stackexchange.com/q/38902` – The set of differences for a set of positive Lebesgue measure.

We have

$$\begin{aligned} D^+v(x) &= \limsup_{y\to x} \frac{v(y)-v(x)}{y-x} \\ &= \limsup_{y\to x} \frac{\int_x^y f(t)\,\mathrm{d}t}{y-x}. \end{aligned}$$

From upper semicontinuity we know that for any given x and $\varepsilon > 0$ we have $f(y) < f(x) + \varepsilon$ for all y's that are close enough to x. So for such y's we get

$$\int_x^y f(t)\,\mathrm{d}t \le (f(x)+\varepsilon)(y-x)$$

and

$$D^+v(x) \le f(x) + \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we finally get $D^+v(x) \le f(x)$. □

Upper function in the proof of Theorem 22.6. In the proof of Theorem 22.6 an upper semicontinuous lower function was obtained directly from Lemma 20.1(ii). The reason the we cannot simply apply dual argument to get a lower semicontinuous upper function is that one of the assumptions of Lemma 20.1(ii) is $f \ge 0$. So the dual argument only works if f is bounded – which is exactly what is used in the following steps of the proof.

Monotonicity and the inequality for D^- in the proof of Theorem 22.6. The fact that $u - (u_1 + \cdots + u_n)$ is non-decreasing is used to get $D^-u \ge D^-(u_1 + \cdots + u_n)$. This is a consequence of the following observation:
If $D^-(f-g) \ge 0$ the $D^-f \ge D^-g$. (Consequently, the same is true if $(f-g)$ is non-decreasing.)

Proof. If suffices to notice that

$$0 \le D^-(f-g) \le D^-f - D^-g.$$

More detailed explanation: $D^-(f-g) \le D^-f + D^+(-g) = D^-f - D^-g$. □

Exercise 22.C. For any class $\mathcal{F}$ of continuous functions on $[a,b]$ one can define $\mathcal{F}$-integrability by calling a function $f\colon [a,b] \to \mathbb{R}$ *$\mathcal{F}$-integrable* if for every $\varepsilon > 0$ f has upper and lower functions $u, v \in \mathcal{F}$ such that

$$\Big|_a^b u \le \Big|_a^b v + \varepsilon.$$

The class of all continuous functions on $[a,b]$ gives us the Perron integral, the class of all absolutely continuous functions defines the Lebesgue integral.

Prove now that each of the following classes of functions leads to the Riemann integral: the class of all piecewise linear continuous functions; the class of all functions that have continuous derivatives.

Solution. Suppose that we have

$$g \le f \le h,$$

where g is upper semicontinuous and h is lower semicontinuous and both g, h are Lebesgue integrable. If we put

$$v(x) = \int_a^x g(s)\,\mathrm{d}s, \qquad u(x) = \int_a^x h(s)\,\mathrm{d}s$$

then from Exercise 22.A we have $D^+v \le g$ and $D^-u \ge h$, hence

$$D^+v \le f \le D^-u.$$

On the other hand, if D^+v and D^-u are Riemann integrable, then we know from Theorem 15.3 that

$$v(x) = \int_a^x D^+v(s)\,\mathrm{d}s, \qquad u(x) = \int_a^x D^-u(s)\,\mathrm{d}s.$$

In fact, here we will be using very simple functions in place of g, h, u, v; so the above can be checked for these functions using direct computation. But since we already know more general results, I have simply mentioned them instead of including the computations.

The basic idea is to use a correspondence between:

- piecewise linear upper functions of f and step functions which are above f;
- continuously differentiable upper functions of f and continuous functions which are above f.

$\mathcal{F}$-integrable function is Riemann integrable. Notice that in both cases we have that D^+v and D^-u are Riemann integrable. (In fact, they are continuous with the exception of at most finitely many points.) So we have

$$\Big|_a^b v = \int_a^b D^+v \le \underline{\int_a^b} f(t)\,\mathrm{d}t \le \overline{\int_a^b} f(t)\,\mathrm{d}t \le \int_a^b D^-u = \Big|_a^b u\,.$$

(Here $\underline{\int}$ and $\overline{\int}$ denotes lower and upper Riemann integral.)

From the fact that $\sup\{\big|_a^b v\} = \inf\{\big|_a^b u\}$ (where supremum and infimum are taken only over lower and upper functions from $\mathcal{F}$, respectively) we get that f is Riemann integrable.

Riemann-integrable function is $\mathcal{F}$-integrable. If a function is Riemann integrable, we can approximate it from below and from above by step functions g and h in such way that $g \le f \le h$ and the function g and h can be made arbitrarily close in the sense

$$\int_a^b (h(t) - g(t))\,\mathrm{d}t < \varepsilon.$$

Notice that we can arbitrarily change values in the points belonging to partition, since we assume that $h(t) \geq \sup_{t \in [a_i, a_{i+1}]} f(t)$ on each interval from the partition (and similarly for g). So we can also assume that g and h are upper and lower semicontinuous, respectively. Therefore we get

$$D^+ v \leq g \leq f \leq h \leq D^- u$$

for v and u defined as above. Moreover, v and u are piecewise linear and we have

$$\int_a^b g(t)\,\mathrm{d}t = \big|_a^b v \qquad \int_a^b h(t)\,\mathrm{d}t = \big|_a^b u\,.$$

So in this way we get the f is $\mathcal{F}$-integrable if $\mathcal{F}$ is the class of piecewise linear continuous functions.

For C^1-functions basically the same argument works, but we need to show that Riemann integrable function f can be approximated by continuous functions h and g. To achieve this we simply modify the step functions by adding small enough triangles. In this way we get g and h which are piecewise linear continuous functions and their indefinite integrals u and v are C^1.

Remark. We could get in a similar way C^∞-functions. Instead of a small triangle we would use some kind of mollifier. □

Exercise 22.D. Show that F need not be absolutely continuous.

Here

$$F(x) = \mathcal{P} \int_a^x f$$

for a Perron integrable function $f(x)$.

Solution. See (8) in introduction. We have a function $H(x)$ (0.2) which has a derivative $h(x)$ given by (0.1).

Since $h(x)$ has an antiderivative, it is Perron integrable by Theorem 22.2.

However, the function $H(x)$ is not absolutely continuous. (It does not have bounded variation.)

Another function which is differentiable, but does not have bounded variation is given in Exercise 3.F. It is the function $x \mapsto x^2 \sin \frac{1}{x^2}$. □

Exercise 22.E. If $f\colon [a,b] \to \mathbb{R}$ is Perron integrable and if $|f| \leq g$ for some Perron integrable $g\colon [a,b] \to \mathbb{R}$, then f is Lebesgue integrable.

Solution. We have $|f| \leq g$, which means

$$-g \leq f \leq g.$$

Therefore the differences $g - f$ and $f - (-g) = f + g$ are both non-negative and Perron integrable. By Corollary 22.8 this implies that both these functions are Lebesgue integrable and therefore also

$$f = \frac{(f-g) + (f+g)}{2}$$

is Lebesgue integrable. □

Exercise 22.F. Prove analogues of the convergence theorems of Levi, Fatou and Lebesgue (19.13, 19.14, 19.15) for the Perron integral.

Exercise 22.F - Levi. Let $f_1 \le f_2 \le \dots$ a.e., where each f_i is Perron integrable. Let the sequence $\mathcal{P}\int_a^b f_1, \mathcal{P}\int_a^b f_2, \dots$ be bounded. Then the sequence $f_1, f_2, \dots$ converges a.e. If $f\colon \mathbb{R} \to \mathbb{R}$ and if $f = \lim\limits_{n\to\infty} f_n$ a.e., then f is Perron integrable and $\mathcal{P}\int_a^b f = \lim\limits_{n\to\infty} \mathcal{P}\int_a^b f_n$.

Proof. It suffices to apply monotone convergence theorem for Lebesgue integral (Theorem 19.13) to the functions $f_k - f_1$. Since $f_k - f_1 \ge 0$ and this function is Perron integrable, it is also Lebesgue integrable (Corollary 22.8). □

Exercise 22.F - Fatou. Directly in the statement of Fatou's lemma we have $f_n \ge 0$. For non-negative functions Lebesgue integral and Perron integral are equivalent.

Exercise 22.F - Lebesgue. Let g be Perron integrable and $g \ge 0$ a.e. Let $f_1, f_2, \dots$ be Perron integrable functions such that $|f_n| \le g$ a.e for all n. Let $f\colon [a,b] \to \mathbb{R}$, $f = \lim\limits_{n\to\infty} f_n$ a.e. Then f is Perron integrable and $\mathcal{P}\int_a^b f = \lim\limits_{n\to\infty} \mathcal{P}\int_a^b f_n$.

Since $g \ge 0$, we get that g is Lebegue integrable (Corollary 22.8). Since $|f_n| \le g$, the functions f_n are also Lebesgue integrable (Exercises 22.E.) Now we can use dominate convergence theorem for Lebesgue integral (Theorem 19.15).

Exercise 22.G. Find a Perron integrable function on $[a,b]$ whose absolute value is not Perron integrable.

Solution. It suffices to take any function which is Perron integrable but not Lebesgue integrable. (Some examples of such functions are given in Exercise 22.D.)

If both f and $|f|$ are Perron integrable, then applying Exercise 22.E with $g = |f|$ we get that f is Lebesgue integrable. □

6.23 The Stieltjes integral

Exercise 23.I. Let $x_1, x_2, \dots \in [a,b]$. Let $p_1, p_2, \dots \in \mathbb{R}$, $\sum |p_i| < \infty$. Show that there exists a function g on $[a,b]$ such that

$$\sum_{n=1}^{\infty} p_n f(x_n) = \int_a^b f \,\mathrm{d}g$$

for $f \in \mathscr{C}[a,b]$.

Exercise 23.J. Let $g\colon [a,b] \to \mathbb{R}$ be increasing, $A = g(a)$, $B = g(b)$. Define $h\colon [A,B] \to [a,b]$ by

$$h(y) := \inf\{x \in [a,b]; g(x) \geq y\} \qquad (y \in [A,B])$$

h is increasing and $g(h(y)) = y$ for every continuity point of y of h. Show that

$$\int_a^b f(g(x))\,\mathrm{d}x = \int_A^B f\,\mathrm{d}h \tag{*}$$

for every continuous function f on $[A,B]$. (Prove $(*)$ first for $f = \xi_{[y,B]}$ where y is a continuity point of h. Now use the techniques of Exercise 23.H.)

Properties of $h(y)$. The function h is non-decreasing, i.e.

$$y_1 \leq y_2 \Rightarrow h(y_1) \leq h(y_2).$$

To see this, just notice that $\{x; g(y) \geq y_2\} \subseteq \{x; g(y) \geq y_1\}$.

It is also easy to see that

$$x < h(y) \Rightarrow g(x) < y$$
$$x > h(y) \Rightarrow g(x) \geq y$$

TODO $g(h(y)) = y$ for continuity points of h.

□

Solution. TODO □

Exercise 23.L. Let $g\colon [a,b] \to [A,B]$ be Lebesgue integrable. For $x \in [A,B]$, let $h(x)$ denote the Lebesgue measure of the set $\{t \in [a,b]; g(t) \leq x\}$. Then h is increasing. Let χ be the function $x \mapsto x$ $(x \in [A,B])$. The Stieltjes integral $\int_A^B \chi\,\mathrm{d}h$ exists and is equal to the Lebesgue integral $\int_a^b g$. (Thus the Lebesgue integral of a bounded function can be expressed in terms of Stieltjes integral and the Lebesgue measure.)

$$h(x) = \lambda(\{t \in [a,b]; g(t) \leq x\})$$

Solution using integration by parts. Since h is monotone it is also Riemann integrable. TODO

Therefore the integral $\int_A^B h\,\mathrm{d}\chi$ exists. (This is precisely the Riemann integral of $h(x)$ by Theorem 23.5.)

Using integration by parts (Theorem 23.6) we get that also $\int_A^B \chi\,\mathrm{d}h$ exists and that

$$\int_A^B \chi\,\mathrm{d}h + \int_A^B h\,\mathrm{d}\chi = B(b-a) - Ah(A).$$

So it suffices to calculate the integral $\int_A^B h\,\mathrm{d}\chi$, i.e., the Riemann integral of h.

TODO Riemann sequence consisting of $A = x_0 < \cdots < x_n = B$ with ξ_i being the right endpoint in each interval.

$$\begin{aligned}
S_V(h,\chi) &= \sum_{i=0}^{n-1}(x_{i+1}-x_i)\lambda(\{t; g(t) \le x_{i+1}\}) \\
&= \sum_{i=0}^{n-1} x_{i+1}\lambda(\{t; g(t) \le x_{i+1}\}) - \sum_{i=0}^{n-1} x_i\lambda(\{t; g(t) \le x_{i+1}\}) \\
&= \sum_{i=1}^{n} x_i\lambda(\{t; g(t) \le x_i\}) - \sum_{i=0}^{n-1} x_i\lambda(\{t; g(t) \le x_{i+1}\}) \\
&= -\sum_{i=1}^{n-1} x_{i+1}\lambda(\{t; x_i < g(t) \le x_{i+1}\}) - x_0\lambda(\{t; g(t) \le x_0\}) + x_n\lambda(\{t; g(t) \le x_n\}) \\
&= -\int_a^b \sum_{i=0}^{n-1} x_{i+1}\xi_{\{t; x_i < g(t) \le x_{i+1}\}} - Ah(A) + B(b-a)
\end{aligned}$$

We may also notice that the function

$$t(x) = \sum_{i=0}^{n-1} x_{i+1}\xi_{\{t; x_i < g(t) \le x_{i+1}\}}(x)$$

is an upper estimate for the function g. (We could also get a similar lower estimate $s(x) = \sum_{i=0}^{n-1} x_i\xi_{\{t; x_i < g(t) \le x_{i+1}\}}(x)$. It is easy to see that both $s(x)$ and $t(x)$ are Lebesgue integrable, since each $\xi_{\{t; x_i < g(t) \le x_{i+1}\}}$ is Lebesgue integrable.)

Now if we take a sequence (V_k) of partitions with norm $\Delta(V_k)$ converging to zero, the corresponding functions $t_k(x)$ converge monotonically to $f(x)$. In fact, the same is true for $s_k(x)$. It suffices to notice that

$$s_k(x) \le g(x) \le t_k(x)$$

and

$$t_k(x) - s_k(x) \le \delta_k$$

where δ_k is the mesh of the Riemann sequence V_k.

So from Theorem 19.13 (monotone convergence, Levi's theorem – it is applicable here since the sequence $\int_a^b t_k$ is bounded) we get that $S_V(h,\chi)$ converges to

$$-\int_a^b g - Ah(A) + B(b-a).$$

Combining the equalities

$$\int_A^B \chi \,\mathrm{d}h + \int_A^B h \,\mathrm{d}\chi = B(b-a) - Ah(A)$$

$$\int_A^B h \,\mathrm{d}\chi = -\int_a^b g - Ah(A) + B(b-a)$$

we get

$$\int_A^B \chi \, \mathrm{d}h = \int_a^b g.$$

□

Exercise 23.M. There does not exist a function g on $[0,1]$ such that

$$f'(0) = \int_0^1 f \, \mathrm{d}g$$

for all polynomial functions on $[0,1]$. (Hint. Suppose that such g exists. Deduce from Theorems 23.6 and 23.5 that g is Riemann integrable. Show that $-h(0) = \int_0^1 h(x)(g(x) - g(1)) \, \mathrm{d}x$ for all polynomial functions h on $[0,1]$, hence for all $h \in \mathscr{C}[0,1]$. Derive a contradiction.)

Deriving (6.1) *and* (6.2). TODO □

So we have that the following is true for any continuous function $h(x)$.

$$\int_0^1 h(x)(g(x) - g(1)) \, \mathrm{d}x = -h(0) \tag{6.1}$$

$$h(0) + \int_0^1 g(x)h(x) \, \mathrm{d}x = g(1)h(x) \, \mathrm{d}x \tag{6.2}$$

Getting a contradiction from (6.1) *and* (6.2). TODO □

Appendixes

A. The real number system

B. Cardinalities

C. An uncountable well-ordered set: a characterization of the functions of the first class of Baire

D. An elementary proof of Lebesgue's density theorem

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