Blümlinger: Lévy group action and invariant measures on $\beta\mathbb{N}$

Notes from [B]

Introduction

For $f \in \ell_\infty$ we put $Tf(n) = \frac{1}{n}\sum_{i=1}^{n} f(i)$.

Any element f of $\ell^\infty(\mathbb{N})$ has a unique continuous extension to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. The continuous extension of χ_A (where $A \subseteq \mathbb{N}$) is $\chi_{\overline{A}}$ (closure in $\beta\mathbb{N}$). We will denote the continuous extension of f to $\beta\mathbb{N}$ again by f. This isometry between $\ell^\infty(\mathbb{N})$ and $C(\beta\mathbb{N})$ allows us to express continuous functions on $\beta\mathbb{N}$ by their values on integers, i.e. as bounded sequence on $\mathbb{N}$, to interpret T as a continuous operator on $C(\beta\mathbb{N})$ and to identify the dual spaces of $\ell^\infty(\mathbb{N})$ (finitely additive measures on $\mathbb{N}$) and of $C(\beta\mathbb{N})$ (regular Borel measures on $\beta\mathbb{N}$).

We identify the natural numbers with their neighborhood filters, so $\mathbb{N} \subset \beta\mathbb{N}$. A base for the topology on $\beta\mathbb{N}$ is now given by the clopen sets. A subset of $\beta\mathbb{N}$ is clopen iff it is the closure $\overline{A}$ of a subset A of $\mathbb{N}$, i.e. iff it is the set of all ultrafilters containing A. With this abuse of notation we can see $A \subset \mathbb{N}$ both as a subset of $\beta\mathbb{N}$ and as an element of an ultrafilter p in $\overline{A}$, i.e. we have $p \in \overline{A}$ iff $A \in p$. See [W] for a detailed exposition.

S permutation g of $\mathbb{N}$ acts naturally as an automorphism on $\beta\mathbb{N}$. For $p \in \beta\mathbb{N}$ let the ultrafilter gp be defined as $gp = \{gA; A \in p\}$.

A permutation g of $\mathbb{N}$ also induces the isometry of $\ell^\infty(\mathbb{N})$: $f \mapsto f_g$, $f_g(n) = f(gn)$ and an isometry of $C(\beta\mathbb{N})$: $f \mapsto f_g$, $f_g(p) = f(gp)$.

TODO??? What the following symbols mean? $\langle \mathcal{M}, C(\beta\mathbb{N})\rangle$, μ_g

Permutation groups and invariant means on $\mathbb{N}$

Lemma. *(Lemma 1) For a group G of permutations of $\mathbb{N}$ the following are equivalent:*

(a) *There exists a G-invariant mean on $\mathbb{N}$;*

(b) *There exists a nonzero G-invariant finitely additive measure on $\beta\mathbb{N}$;*

(c) *There exists a G-invariant probability mesasure on $\beta\mathbb{N}$;*

(d) *There exists a nonzero G-invariant (finite Borel) on $\beta\mathbb{N}$;*

G-invariant measures on $\beta\mathbb{N}$ correspond to G-invariant finitely additive measures on $\mathbb{N}$

$g \in \mathcal{G}_\delta \Leftrightarrow \lim\limits_{n\to\infty} Tf(n) - Tf_g(n) = \lim\limits_{n\to\infty} Tf(n) - Tf_{g^{-1}}(n) = 0$ for all f in the subspace of Cesaro summable sequences in $\ell_\infty \Leftrightarrow \lim\limits_{n\to\infty} d_A(n) - d_{gA}(n) = \lim\limits_{n\to\infty} d_A(n) - d_{g^{-1}A}(n) = 0$ for all subsets A of $\mathbb{N}$ which have density.

The following are equivalent:

(i) $g \in \mathcal{G}$,

(ii) $\forall f \in \ell_\infty \lim_{n\to\infty} Tf(n) - Tf_g(n) = 0$,

(iii) $\forall A \subset \mathbb{N} \lim_{n\to\infty} d_A(n) - d_{gA}(n) = 0$.

Lema 3: Let $A, B \subset \mathbb{N}$ such that A, B, A^c, B^c are infinite sets. Then there is a $g \in \mathcal{G}$ with $B = gA$ if and only if $\lim_{n\to\infty} d_A(n) - d_B(n) = 0$.

There is no nontrivial $\mathcal{G}_\delta$-invariant functional on ℓ_∞.

References

[B] M. Blümlinger. Lévy group action and invariant measures on $\beta\mathbb{N}$. *Trans. Amer. Math. Soc.*, 348(12):5087–5111, 1996.

[W] R. C. Walker. *The Stone-Čech compactification.* Springer-Verlag, Berlin, Heidelberg, New York, 1974.