What is amenable group?
$d =$ asymptotic density
$aut(\mathbb{N}) =$ permutations of $\mathbb{N}$
$\mathcal{G} = \{\pi \in aut(\mathbb{N}); \lim_{N\to\infty} |\{n \leq N < \pi(n)\}|\} =$ Lévy group

Amenable group

`planetmath.org`: Let G be a locally compact group and $L^\infty(G)$ be the space of all essentially bounded functions $G \to \mathbb{R}$ with respect to the Haar measure.

A linear functional on $L^\infty(G)$ is called a *mean* if it maps the constant function $f(g) = 1$ to 1 and non-negative functions to non-negative numbers.

Let L_g be the left action of $g \in G$ on $f \in L^\infty(G)$, i.e. $(L_g f)(h) = f(g^{-1}h)$. Then, a mean μ is said to be *left invariant* if $\mu(L_g f) = \mu(f)$ for all $g \in G$ and $f \in L^\infty(G)$. Similarly, *right invariant* if $\mu(R_g f) = \mu(f)$, where R_g is the right action $(R_g f)(h) = f(gh)$.

A locally compact group G is *amenable* if there is a left (or right) invariant mean on $L^\infty(G)$.

All finite groups and all abelian groups are amenable. Compact groups are amenable as the Haar measure is an (unique) invariant mean.

If a group contains a free (non-abelian) subgroup on two generators then it is not amenable.

`wolfram`: If a group contains a (non-abelian) free subgroup on two generators, then it is not amenable. The converse to this statement is the Von Neumann conjecture, which was disproved in 1980.

N. Obata: Density of Natural Numbers and Lévy Group

$\mathcal{F} =$ sets with density

Darboux property of asymptotic density is shown here.
$\mathcal{G}(d) =$ permutations which preserve density
$\mathcal{G}_0 = \{\pi : d(\{n : \pi(n) \neq n\}) = 0\}$
$\mathcal{G}_0 \subset \mathcal{G} \subset \mathcal{G}(d)$
All inclusions are proper. $\mathcal{G}_0$ is a normal subgroup $\mathcal{G}(d)$.

Proposition. *Let A and B be members of $\mathcal{F}$ such that $0 < d(A) = d(B) < 1$. Then there exists a permutation $g \in \mathcal{G}$ such that $g(A) = B$.*

Theorem. *Assume that $A \in \mathcal{F}$ is almost invariant under the Lévy group, i.e.,*

$$d(A \triangle g(A)) = 0 \quad \textit{for all } g \in \mathcal{G}.$$

Then $d(A) = 0$ or $d(A) = 1$.

Finite additivity, invariance with respect to Lévy group and normalization ($d(\mathbb{N}) = 1$) characterize density. (Theorem 2)

$$L^+ = \limsup_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n, \qquad L^- = \liminf_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n$$

Proposition. *The Lévy group is the maximal permutation group which keeps L^+ (or L^-) invariant.*

A sequence (x_n), $0 \le x_n < 1$ is *uniformly distributed* on the interval $\langle 0, 1)$ if

$$\lim_{N\to\infty} \frac{1}{N} |\{1 \le n \le N; a \le x_n < b\}| = b - a$$

for every $a, b \in \langle 0, 1\rangle$, $a < b$.

$a = (a_n) \mapsto ga = (a_{g^{-1}(n)})$

If x is uniformly distributed then gx is uniformly distributed as well.

Blümlinger: Lévy group action and invariant measures on $\beta\mathbb{N}$

Further notes can be found in the file `blumlinger.tex`

For $f \in \ell_\infty$ we put $Tf(n) = \frac{1}{n}\sum_{i=1}^{n} f(i)$.

$f_g(n) = f(gn)$

$g \in \mathcal{G}_\delta \Leftrightarrow \lim\limits_{n\to\infty} Tf(n) - Tf_g(n) = \lim\limits_{n\to\infty} Tf(n) - Tf_{g^{-1}}(n) = 0$ for all f in the subspace of Cesaro summable sequences in $\ell_\infty \Leftrightarrow \lim\limits_{n\to\infty} d_A(n) - d_{gA}(n) = \lim\limits_{n\to\infty} d_A(n) - d_{g^{-1}A}(n) = 0$ for all subsets A of $\mathbb{N}$ which have density.

The following are equivalent:

(i) $g \in \mathcal{G}$,

(ii) $\forall f \in \ell_\infty \lim\limits_{n\to\infty} Tf(n) - Tf_g(n) = 0$,

(iii) $\forall A \subset \mathbb{N} \lim\limits_{n\to\infty} d_A(n) - d_{gA}(n) = 0$.

Lema 3: Let $A, B \subset \mathbb{N}$ such that A, B, A^c, B^c are infinite sets. Then there is a $g \in \mathcal{G}$ with $B = gA$ if and only if $\lim\limits_{n\to\infty} d_A(n) - d_B(n) = 0$.

There is no nontrivial $\mathcal{G}_\delta$-invariant functional on ℓ_∞.

Blümlinger, Obata: Permutations preserving Cesaro mean, densities of natural numbers and uniform distribution of sequences

L = Cesaro mean, $\mathcal{D}$=Cesaro summable bounded sequences

Theorem. *A $\mathcal{G}$-invariant positive functional on the space of all Cesaro summable bounded real sequences is a constant multiple of Cesaro mean. A $\mathcal{G}$-invariant positive normalized functional on ℓ_∞ is a Banach limit.*

Lemma. *Let $0 = N_0 < N_1 < \dots$ be an increasing sequence such that $\lim\limits_{k\to\infty} \frac{N_k}{N_{k-1}} = 1$. Then a permutation g on $\mathbb{N}$ which leaves every subset $\{N_k+1,\dots,N_k\}$ invariant belongs to $\mathcal{G}$.*

Let M denote $\mathcal{G}$-invariant continuous functional on $\mathcal{D}$.

Lemma. *Assume that $a = (a_n) \in \mathcal{D}$ satisfies $a_n \in \{0,1\}$, $0 < L(a) < 1$ and $L(a) \in \mathbb{Q}$. Them $M(a) = L(a).M(\overline{1})$.*

I think, that the proof of the above lemma can be applied to show the same claim about measures on $\mathcal{P}(\mathbb{N})$: If $A \in \mathbb{N}$ has $d(A) \in \mathbb{Q}$, then $\mu(A) = d(A)$ for any $\mathcal{G}$-invariant measure on $\mathbb{N}$.

Lemma. *$M(a) = M(\overline{1}).L(a)$ for all $a \in \mathcal{D}$*

Theorem. *There exists a positive continuous linear functional on ℓ_∞ which is invariant under the Lévy group.*

Rao: Theory of Charges

Notation: $I_A = \chi_A$, $\mathcal{L}(\mathcal{F})$ = Def 3.1.1, field on Ω = algebra of sets

Blass, Frankiewicz, Plebanek, Ryll-Nardzewski: A Note on Extensions of Asymptotic density

$\mathcal{F}$ is σ-algebra on a set X, ν is finitely additive measure on $\mathcal{F}$. Say that ν has the property AP(null) if for every increasing sequence $(A_i) \subset \mathcal{F}$ there exists a set $B \in \mathcal{F}$ such that:
(i) $\nu(A_i \setminus B) = 0$ for every i,
(ii) $\nu(B) = \lim \nu(A_i)$.

Given an infinite set $X \subseteq \omega$ we write

$$I_n^X = [\max(X \cap n), n) \cap \omega,$$

whenever $n \in X$. Say that a set X is *thin* if

$$\lim_{n\in X} \frac{|I_n^X|}{n} = 1.$$

In other words, a set X is thin if enumerating X as $(n_k)_k$ in increasing order, we have $\lim\limits_{k\to\infty} \frac{n_k}{n_{k+1}} = 0$.

If an ultrafilter $\mathcal{U}$ contains a thin set then $\nu_{\mathcal{U}}$ is a density having the property AP(null).

Pólya: Untersuchungen über Lücken und Singularitäten von Potenzreihen

Untere und Obere Dichte zur Basis ξ, 1

Eine Folge (λ_n) mit $\lambda_1 \geq 0$ und $\lambda_{n+1} - \lambda_n \geq p > 0$.

Wenn der Grenzwert

$$\lim_{r\to\infty} \frac{N(r)}{r}$$

existier, wird er als *Dichte* der Folge (λ_n) gennant. *meßbare Folge*

$$\liminf_{r\to\infty} \frac{N(r) - N(r\xi)}{r - r\xi} = d(\xi), \qquad \limsup_{r\to\infty} \frac{N(r) - N(r\xi)}{r - r\xi} = D(\xi)$$

bei festem $0 \leq \xi < 1$

Minimaldichte und Maximaldichte

Theorem (Satz III). *Es existieren die Grenzwerte*

$$\lim_{\xi\to 1-0} d(\xi) = d(1), \qquad \lim_{\xi\to 1-0} D(\xi) = D(1)$$

die Minimaldichte und *die Maximaldichte*

Additive Eigenschaften

Theorem (Satz VIII). *Die Minimaldichte einer Folge (λ_n) ist die Dichte der dichtesten meßbaren Folge, die in (λ_n) als Teilfolge enthalten ist, und die Maximaldichte von (λ_n) ist die Dichte der dünnsten meßbaren Folge die (λ_n) als Teilfolge enthält.*

Other

R.C.Buck: Generalized Asymptotic Density In connection with this paper Atila had a talk on our seminar about Caratheodory-like construction for finitely additive measures. Unfortunately, I didn't noted reference in my notes.

$A \dot{\subset} B$ iff $B - A$ is bounded

p.571–572:

We can define outer density $\omega(S)$ as $\inf D(A)$ taken over all sets A in $\mathcal{D}$ which contain S. Pólya has proved that this outer density can be analytically expressed, and is in fact the Pólya maximum density; that is,

$$\omega(S) = \overline{D}_1(S) = \lim_{\theta\to 1^-} \limsup_{n\to\infty} \frac{S(n) - S(\theta n)}{n - \theta n}.$$

More precisely, Pólya proved that if $\overline{D}_1(S) = d$, then there exists a set B of $\mathcal{D}$ having density d and containing S. [**?**, 562]

$\underline{D}_1(S) + \overline{D}_1(S) = 1$

$\mathcal{D}_\omega =$ sets with $\omega(S) + \omega(S') = 1$.

Theorem. $\mathcal{D}_\omega = \mathcal{D}$

(For Pólya maximum density see also:
Koosis: The Logarithmic Integral, Young: An Introduction to Nonharmonic Fourier Series and Wavelet Expansions)

Let $A_1 \dot{\subset} A_2 \dot{\subset} \dots$ and let $\lim \overline{d}(A_n) = \Delta$ and $\lim \underline{d}(A_n) = \delta$. Then, there exists a set A with $\overline{d}(A) = \Delta$ and $\underline{d}(A) = \delta$, such that $A_n \dot{\subset} A$ for all n.

If the sets A_n have density and $A_1 \dot{\subset} A_2 \dot{\subset} \dots$ then there is a set A, unique up to set of zero density, such that $A_n \dot{\subset} A$ for all n and $d(A) = \lim d(A_n)$.

If $C_1, C_2, \dots$ are disjoint sets having density, there is a set C, unique up to set of zero density, such that $C \dot{\supset} \bigcup_{k=1}^n C_k$ for all n and with $d(C) = \sum d(C_k)$.

R.C.Buck: The measure theoretic approach to density