Meyer Jerison: The set of all generalized limits of bounded sequences

This note basically contains some results and proofs from [J2]. (This was made in preparation for a talk on a seminar where most of the participants are used to work with $\mathcal{F}$-limits, therefore it seemed to be suitable to use ultralimits instead of subnets in some places. I also tried to use more modern notation and I used a different representation of $\beta\mathbb{N}$ – it is described as the set of all extreme points of positive normed functionals with the topology induced by the weak* topology in [J2]. Some of the proofs are described in more detail than in the original paper.)

1 Preliminaries

Let us recall some results which will be needed in the proof of the main result. (We will formulate them in the form suitable for our situation.)

1.1 $\mathcal{F}$-limits

We will mostly work with limits along ultrafilters, but $\mathcal{F}$-limit can be defined for any filter $\mathcal{F}$.

Definition 1. A filter on a set M is a non-empty family $\mathcal{F} \subseteq \mathcal{P}(M)$ such that:

(i) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(ii) $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$;

(iii) $\emptyset \notin \mathcal{F}$.

A filter is called *free* if $\bigcap \mathcal{F} = \emptyset$.

A filter can be considered as a criterion saying which sets will be considered large. A simple example of filter is the set of all subsets of $\mathbb{N}$ with finite complements. This filter is called *Fréchet filter.*

Definition 2. A filter $\mathcal{F}$ on M is an *ultrafilter* if for any $A \subseteq M$

$$A \in \mathcal{F} \qquad \vee \qquad M \setminus A \in \mathcal{F}.$$

Ultrafilters are precisely maximal filters (with respect to inclusion).

For any $m \in M$ the filter $\mathcal{F}_m = \{A \subseteq M; m \in A\}$ is an ultrafilter of M. (Such ultrafilters are called *principal.*)

Using Axiom of Choice (or Zorn Lemma) it can be shown that every system of subsets of M with finite intersection property is contained in an ultrafilter. This also implies existence of free ultrafilters.

Definition 3. If (x_n) is a sequence of elements of a topological space X and $\mathcal{F}$ is a filter on $\mathbb{N}$, then we say that (x_n) is $\mathcal{F}$-convergent to $l \in X$ if for every neighborhood U of l the set

$$A(U) = \{n \in \mathbb{N}; x_n \in U\}$$

belongs to $\mathcal{F}$.

Notation: $\mathcal{F}\text{-}\lim x_n = l$.

We can notice that if we consider a sequence (x_n) as a map $x\colon \mathbb{N} \to X$, then $A(U) = x^{-1}(U)$. This suggest how the notion of $\mathcal{F}$-limit can be generalized to maps on any set.

We will only work with Hausdorff spaces, in these spaces a sequence can have at most one $\mathcal{F}$-limit.

It is easy to see that for Fréchet filter the $\mathcal{F}$-convergence is precisely the usual convergence of sequences.

Another simple example: $\mathcal{F}_m\text{-}\lim x = x_m$.

We will need the following property of ultralimits.

Theorem 1. *Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$ and (x_n) be a sequence of points of a compact space X. Then $\mathcal{F}\text{-}\lim x_n$ exists.*

In particular, $\mathcal{F}\text{-}\lim x_n$ exists for any bounded sequence in $\mathbb{R}$.

$\mathcal{F}$-limits of real sequences have many properties similar to the usual limit – e.g. additivity and multiplicativity. For a fixed sequence (x_n), all possible values of $\mathcal{F}\text{-}\lim x_n$ for free ultrafilters are precisely the cluster points of this sequence. In particular, if a sequence is convergent to a limit l, then for every free ultrafilter $\mathcal{F}$ also $\mathcal{F}\text{-}\lim x_n = l$.

1.2 Stone-Čech compactification of integers

We will briefly remind the notion of the Stone-Čech compactification of a topological space. (Although we will only need it for the discrete countable space.)

Definition 4. Let X be a topological space. Then a compact Hausdorff space βX is *Stone-Čech compactification* of X if there exists embedding $i\colon X \hookrightarrow \beta X$ such that for every continuous map $f\colon X \to K$ to a compact Hausdorff space K there exists a unique continuous extension $\overline{f}\colon \beta X \to K$ fulfilling $\overline{f} \circ i = f$.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "\overline{f}"] \\ & K \end{tikzcd} \tag{1}$$

Let us note that we would get an equivalent definition if we worked with the unit interval $I = \langle 0, 1\rangle$ instead of arbitrary compact Hausdorff space K. We will usually identify X and $i[X]$, so X is a subspace of βX.

The Stone-Čech compactification of X is obviously determined uniquely up to a homeomorphism. It is known the βX exists if and only if X is completely regular.

1.2.1 Stone-Čech compactification of integers

We will work with the Stone-Čech compactification of $\mathbb{N}$ endowed with the discrete topology. It can be shown that $\beta\mathbb{N}$ can be obtained as the topological space on the set $\mathrm{Uft}(\mathbb{N})$ of all ultrafilters on $\mathbb{N}$ with the topology given by the base $\{U_A; A \subseteq \mathbb{N}\}$, where

$$U_A = \{\mathcal{F} \in \mathrm{Uft}(\mathbb{N}); A \in \mathcal{F}\}.$$

The embedding i of $\mathbb{N}$ into $\beta\mathbb{N}$ is (for this construction) given by mapping n to the corresponding principal filter

$$i\colon n \mapsto \mathcal{F}_n = \{B \subseteq \mathbb{N}; n \in B\}.$$

In fact, the basic sets U_A are clopen. (The space $\beta\mathbb{N}$ is zero-dimensional.)
The definition of U_A can be rewritten as

$$\mathcal{F} \in U_A \qquad \Leftrightarrow \qquad A \in \mathcal{F}.$$

The continuous extension of a bounded sequence $x\colon \mathbb{N} \to \mathbb{R}$ to $\beta\mathbb{N}$ is given by

$$\overline{x}(\mathcal{F}) = \mathcal{F}\text{-}\lim x,$$

which shows that $\mathcal{F}$-limits and this construction of $\beta\mathbb{N}$ are closely related.

1.2.2 $C(\beta\mathbb{N})$ and $\ell_\infty(\mathbb{N})$

This part should show why the space $\beta\mathbb{N}$ could be interesting for us when we are studying ℓ_∞.

Proposition 1. *Banach spaces ℓ_∞ and $C(\beta\mathbb{N})$ are isometrically isomorphic, i.e., there exists a norm-preserving linear isomorphism between them.*

Proof. We can define map $\varphi\colon \ell_\infty \to C(\beta\mathbb{N})$

$$\varphi(x) = \overline{x}$$

which assigns to each bounded sequence x its continuous extension on $\beta\mathbb{N}$.
In the other way we can use the restriction from $\beta\mathbb{N}$ to $\mathbb{N}$ $\psi\colon C(\beta\mathbb{N}) \to \ell_\infty$

$$\psi(f) = f \circ i$$

where $i\colon \mathbb{N} \hookrightarrow \beta\mathbb{N}$ is the embedding of $\mathbb{N}$ into $\beta\mathbb{N}$. (If we identify $\mathbb{N}$ with the corresponding subspace of $\beta\mathbb{N}$, this is the same as $f \mapsto f|_{\mathbb{N}}$.)
It is easy to show that φ and ψ are inverse to each other, linear and norm-preserving. □

Corollary 1. *Banach space ℓ_∞^* and $C^*(\beta\mathbb{N})$ are isometrically isomorphic.*
The spaces ℓ_∞^* and $C^*(\beta\mathbb{N})$ endowed with the weak-topologies are isomorphic (in the sense that there exists linear homeomorphism between the two topological vector spaces).*

1.3 Banach-Alaoglu theorem

We will often work with the weak*-topology on X^*, where X is a linear normed space. (In fact, we will work with $X = \ell_\infty$.)

For every $x \in X$ we have a linear map $x^* \colon X^* \to \mathbb{R}$ defined by

$$x^*(f) = f(x).$$

These maps are continuous with respect to norm-topology on X^*.

Definition 5. The *weak*-topology* is the weakest topology on X^* such that all maps x^* are continuous with respect to this topology.

The weak* topology is in fact the topology induced on X^* considered as a subspace as the topological product $\mathbb{R}^X$.

The weak*-topology can be equivalently described using nets: A net $(f_d)_{d\in D}$ is convergent to f in the weak*-topology if and only if the net $(f_d(x))_{d\in D}$ converges to $f(x)$ for each $x \in X$.

An important property of the weak*-topology is the following:

Theorem 2 (Banach-Alaoglu)**.** *Let $B = \{f \in X^*; \|f\| \leq 1\}$ be the unit ball of X^*. The set B is compact in the weak*-topology on X^*.*

1.4 Krein-Milman theorem

Krein-Milman theorem describes closed convex subset of a locally convex topological vector space using its extreme points. We do not want to include details about locally convex TVS – interested reader can find them in most functional analytic textbooks or in texts devoted to topological vectors spaces. Let us just mention that if X is a linear normed space, then X with the norm-topology is locally convex TVS and X^* with the weak*-topology is a locally convex space.

Definition 6. Let C be a subset of a TVS E. Let $e \in C$. The point e is an *extreme point* of the set C if for $x_{1,2} \in C$

$$\frac{x_1 + x_2}{2} = e \qquad \Rightarrow \qquad x_1 = x_2 = e$$

holds. In the other words: The point e cannot be expressed as a non-trivial convex combination of points from C.

The notion of extreme point is illustrated in Figure 1 (taken from [WIK]).

We will use the following formulation of Krein-Milman theorem:

Theorem 3 ([J1, Theorem 1])**.** *Let E be a locally convex TVS and C be a compact convex subset of E. Let $S \subseteq C$. The following assertions are equivalent:*

(i) For every linear continuous function $f \colon E \to \mathbb{R}$ the equality

$$\sup_{x\in S} f(x) = \sup_{x\in C} f(x)$$

holds;

(ii) $C = \overline{\mathrm{co}}(S)$, i.e. C is the closed convex hull S;

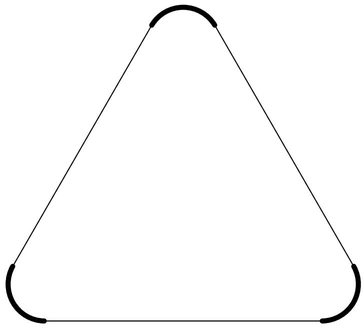

Figure 1: Extreme points of a set

(iii) the closure $\overline{S}$ of the set S contains all extreme points of C.

From the above result we can get the usual formulation of Krein-Milman theorem.

Corollary 2 (Krein-Milman)**.** *Let E be a locally convex TVS and C be a compact convex subset of E. Then C is the closed convex hull of the set of all extreme points of C.*

$$C = \overline{\mathrm{co}}(\mathrm{Ext}(C))$$

We will use Theorem 3 in the special case where E is X^* with the weak* topology. It is known that in this case continuous linear functionals on E are precisely the maps x^* for $x \in X$ (see [FHH$^+$, Proposition 3.22]). Hence in this case we get:

Proposition 2. *Let X be a linear normed space and C be a subset of X^* which is convex and compact in the weak*-topology. Let $S \subseteq C$. The following conditions are equivalent:*

(i)

$$\sup_{\varphi \in S} \varphi(x) = \sup_{\varphi \in C} \varphi(x) \tag{2}$$

holds for each $x \in X$;

(ii) $C = \overline{\mathrm{co}}(S)$, i.e. C is the closed convex hull S;

(iii) the closure $\overline{S}$ of the set S contains all extreme points of C.

1.5 Birkhoff's ergodic theorem

Definition 7. A quadruple $(X, \mathcal{B}, \mu, T)$ is a measure preserving system if $\mathcal{B}$ is a σ-algebra on X, μ is a measure on $\mathcal{B}$ and map $T\colon X \to X$ is measurable and fulfills the condition

$$\mu(T^{-1}A) = \mu(A)$$

for each $A \in \mathcal{B}$.

We will use the following result, which is known as Birkhoff's ergodic theorem (or pointwise ergodic theorem or individual ergodic theorem).

Theorem 4 ([EW, Theorem 2.30])**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. If $f \in L_1(\mu)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = f^*(x)$$

converges almost everywhere and in $L_1(\mu)$ to a T-invariant function $f^* \in L_1(\mu)$ and

$$\int f^* \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

Birkhoff's ergodic theorem also claims that f^* is constant (almost everywhere) if T is ergodic, but we will not need this fact.

For any functions $T\colon X \to X$ and $f\colon X \to \mathbb{R}$ we will use the notation $T_n(f) = \frac{f+fT+\dots+fT^{n-1}}{n}$. I.e., $T_n(f)\colon X \to \mathbb{R}$ is the function given by

$$T_n(f)(x) = \frac{f(x) + f(Tx) + \dots + f(T^{n-1}x)}{n} \tag{3}$$

for any $x \in X$.

Hence the Birkhoff's ergodic theorem says that for every measure preserving transformation the sequences $T_n(f)$ converges pointwise to some f^* and that f^* and f have the same integral.

1.6 Riesz representation theorem

Regular Borel measure is a measure μ on a topological space with the following properties:

- μ is defined on the σ-algebra $\mathcal{B}$ of all Borel sets;
- $\mu(K) < \infty$ for every compact subset;
- $\mu(B) = \sup\{\mu(K); K \subseteq B; K \text{ is compact}\}$ for every $B \in \mathcal{B}$;
- $\mu(B) = \inf\{\mu(U); U \supseteq B; U \text{ is open}\}$ for every $B \in \mathcal{B}$ such that $\mu(B) < \infty$. (If we assume the validity of the last condition for open sets, we get an equivalent definition.)

Theorem 5 (Riesz Representation Theorem, [AB, Theorem 38.3])**.** *Let X be a compact Hausdorff space. For every positive linear functional F on $C(X)$, there exists a unique regular Borel measure μ on X such that*

$$F(f) = \int_X f \, \mathrm{d}\mu$$

holds for every $f \in C(X)$.

We will later use this theorem for the compact space $\beta\mathbb{N}$.

2 Banach limits

We finally arrive to the main point of interest - the notion of Banach limit.

The usual limit can be understood as a linear functional on the space c of all convergent sequences. We want to find some functionals on a larger spaces, which still have some nice properties similar to the usual limit. We will work with linear functional on the space ℓ_∞ of all bounded sequences.

We have already seen one example of such generalizations. If we choose any free ultrafilter $\mathcal{F}$ on $\mathbb{N}$, then $f\colon \ell_\infty \to \mathbb{R}$ given by

$$f\colon x \mapsto \mathcal{F}\text{-}\lim x_n$$

is a linear functional on ℓ_∞ which extends limit, i.e., $f|_c = \lim$.

Another interesting property of this functional is multiplicativity:

$$f(x.y) = \mathcal{F}\text{-}\lim x_n.y_n = \mathcal{F}\text{-}\lim x_n.\,\mathcal{F}\text{-}\lim y_n = f(x).f(y).$$

We would like to study the extensions of limits which fulfill another interesting property – shift-invariance.

Let us define shift-operator $T\colon \ell_\infty \to \ell_\infty$ by

$$T\colon (x_n) \mapsto (x_{n+1}).$$

A functional $f \in \ell_\infty^*$ is said to be *shift-invariant* if

$$f(Tx) = f(x).$$

It is easy to see that an extension of limit cannot be simultaneously shift-invariant and multiplicative. Just take the sequence $x = (1,0,1,0,\dots)$ and notice that $x + Tx = \overline{1}$, which implies that $f(x) + f(Tx) = f(\overline{1}) = 1$

$$f(x) = f(Tx) = \frac{1}{2}$$

for every shift-invariant extension of limit. We also have $x.x = x$, thus for a multiplicative extension of limit we get

$$f(x)^2 = f(x),$$

therefore the only possible values of $f(x)$ for are in this case 0 and 1.

Definition 8. A linear functional $f\colon \ell_\infty \to \mathbb{R}$ is called *Banach limit,* if it is positive, shift-invariant and extends limit, i.e.,

(i) $x \geq 0 \Rightarrow f(x) \geq 0$;

(ii) $(\forall x \in \ell_\infty) f(Tx) = f(x)$;

(iii) if x is convergent then $f(x) = \lim x$.

The above properties imply that

$$\liminf x_n \leq f(x) \leq \limsup x_n$$

holds for each Banach limit. Together with $f(\overline{1}) = 1$ we get that $\|f\| = 1$ for every Banach limit. (In particular, every Banach limit belongs to ℓ_∞^*.)

Although we do not know yet whether any Banach limits exist at all, we might mention the following result at this place.

Lemma 1. *The set of all Banach limits is weak*-closed subset of ℓ_∞^*. Consequently, it is compact in weak* topology.*

The set of all Banach limits is a convex subset of ℓ_∞^*.

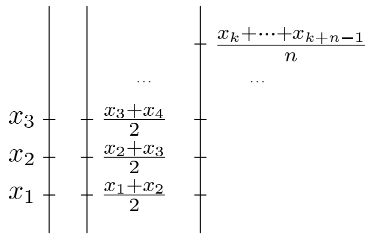

Figure 2: The sequences $T_n(x)$

Proof. Let $(f_d)_{d\in D}$ be a net of elements of ℓ_∞^* which converges pointwise to $f \in \ell_\infty^*$.

It is easy to show that:
If each f_d is positive, then the limit is positive.
If each f_d is shift-invariant then the limit is shift-invariant.
If each f_d extends limit then the limit f extends limit.

Together we get that limit of a net of Banach limits is again a Banach limit. Hence the set of all Banach limits is a weak*-closed subset of unit ball. Hence by Banach-Alaoglu theorem it is also compact.

Convexity can be verified similarly – by verifying that all of the properties from the definition of Banach limit are preserved by convex combinations. □

3 Existence of Banach limit and their extreme values

For any bounded sequence x we define $T_n(x) = \frac{x+Tx+\dots+T^{n-1}x}{n}$. I.e., $T_n(x)$ is the sequence $\left(\frac{x_k+x_{k+1}+\dots+x_{k+n-1}}{n}\right)_{k=1}^{\infty}$ (in fact, if we define $S\colon \mathbb{N}\to\mathbb{N}$, $S\colon n\mapsto n+1$ and $x\colon \mathbb{N}\to\mathbb{R}$, then $T_n(x)$ is the same thing as $S_n(x)$ using the notation from (3).)

The sequences $T_n(x)$ are illustrated in Figure 2.

We define

$$M(x) = \lim_{n\to\infty} \limsup T_n(x)$$
$$m(x) = \lim_{n\to\infty} \liminf T_n(x)$$

The existence of the above limits can be proven by showing that the sequence $a_n = \limsup_{k\to\infty}(x_k+x_{k+1}+\dots+x_{k+n-1})$ is subadditive, i.e. it fulfills $a_{m+n} \le a_m + a_n$ and then use Fekete's lemma, which say that for every subadditive sequence (a_n) the limit $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists and it is equal to $\inf \frac{a_n}{n}$.

Since for every Banach limit f we have $f(x) = f(T_n(x)) \le \limsup T_n(x)$. This implies that the inequalities

$$m(x) \le f(x) \le M(x)$$

are valid for each Banach limit f.

3.1 Proof using Hahn-Banach theorem

Using Hahn-Banach theorem for the functional $\lim\colon c \to \mathbb{R}$ majorized by the function $M(x)$ we get the existence of a linear functional $f\colon \ell_\infty \to \mathbb{R}$ extending limit and fulfilling

$$m(x) \le f(x) \le M(x)$$

for each $x \in \ell_\infty$.

The above conditions clearly imply positivity and (using $m(x-Tx) = M(x-Tx) = 0$) also shift-invariance.

Moreover, it can be shown using Hahn-Banach theorem, that for all values from the interval $\langle m(x), M(x)\rangle$ there exists an extension with these properties.

Proposition 3. *For a given bounded sequence x all possible values of Banach limits are the values from the interval $\langle m(x), M(x)\rangle$.*

This implies that all Banach limits have the same value l for a given sequence if and only if this sequence fulfills $m(x) = M(x) = l$. This is equivalent to

$$\lim_{k\to\infty} \frac{x_k + x_{k+1} + \cdots + x_{k+n-1}}{n} = l$$

uniformly with respect to k. Such sequences are called *almost convergence* and they were characterized by Lorentz [L].

3.2 Proof using ultralimits

Another simple proof of existence of Banach-limits is the following:

Let $\mathcal{F}$ be any free ultrafilter. Let us define $f\colon \ell_\infty \to \mathbb{R}$ as

$$f(x) = \mathcal{F}\text{-}\lim \frac{x_1 + \cdots + x_n}{n}.$$

Using basic properties of $\mathcal{F}$-limits it can be shown that the function f is linear and extends limit. It is also shift-invariant, since

$$f(Tx - x) = \mathcal{F}\text{-}\lim \frac{x_{n+1} - x_1}{n} = 0$$

since $\frac{x_{n+1}-x_1}{n} \le \frac{2\|x\|}{n}$ converges to 0 in the usual sense.

However, Banach limits of this form can only attain values between $\liminf \frac{x_1+\cdots+x_n}{n}$ and $\limsup \frac{x_1+\cdots+x_n}{n}$. We can slightly modify the above proof, e.g. by working with

$$f(x) = \mathcal{F}\text{-}\lim \frac{x_{p_n} + \cdots + x_{p_n+n-1}}{n},$$

where (p_n) is arbitrary integer sequence. The proof that f is a Banach limit is almost the same as in the preceding case. But for these functional we can show that both values $m(x)$ and $M(x)$ can be attained. Using convexity we get that the values attained for a fixed $x \in \ell_\infty$ are precisely the numbers from the interval $\langle m(x), M(x)\rangle$, i.e., using ultralimits we can also prove Proposition 3.

4 Main result

We have already described the extreme values of Banach limits for a fixed sequence. Proposition 2 suggests that this could help us to describe Banach limits as closed convex hulls of some sets.

4.1 Shift on $\beta\mathbb{N}$

We have introduced shift operator $T\colon \ell_\infty \to \ell_\infty$. We want to show that this operator is associated with a continuous function $\overline{S}\colon \beta\mathbb{N} \to \beta\mathbb{N}$ in a natural way.

Let us start with the function $S\colon \mathbb{N} \to \mathbb{N}$ given by $S(n) = n + 1$. Then there exists a continuous function $\overline{S}\colon \beta\mathbb{N} \to \beta\mathbb{N}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "i"] \arrow[d, "S"'] & \beta\mathbb{N} \arrow[d, "\overline{S}"] \\ \mathbb{N} \arrow[r, "i"'] & \beta\mathbb{N} \end{tikzcd} \tag{4}$$

(The function $\overline{S}$ is the unique extension of $i \circ S$ from the definition of Stone-Čech compactification.)

Suppose now that $x\colon \mathbb{N} \to \mathbb{R}$ is a bounded sequence. Then $\overline{x \circ S}$ is the unique continuous map $\overline{x \circ S}\colon \beta\mathbb{N} \to \mathbb{R}$ such that

$$\overline{x \circ S} \circ i = x \circ S.$$

$$\begin{tikzcd} \mathbb{N} \arrow[r, "i"] \arrow[d, "S"'] & \beta\mathbb{N} \arrow[d, "\overline{x \circ S}"] \\ \mathbb{N} \arrow[r, "x"'] & \mathbb{R} \end{tikzcd}$$

From the definition of Stone-Čech compactification (more precisely, from the definition of $\overline{S}$ and $\overline{x}$) we also get

$$\overline{x} \circ \overline{S} \circ i = \overline{x} \circ i \circ S = x \circ S.$$

So the map $\overline{x} \circ \overline{S}$ fulfills the condition that uniquely determines $\overline{x \circ S}$, which means

$$\overline{x \circ S} = \overline{x} \circ \overline{S}. \tag{5}$$

This implies that the map $\overline{S}\colon \beta\mathbb{N} \to \beta\mathbb{N}$ fulfills

$$\overline{x} \circ \overline{S}(\mathcal{F}) = \overline{x \circ S}(\mathcal{F}) = \overline{Tx}(\mathcal{F}) = \mathcal{F}\text{-}\lim Tx.$$

(We have used $Tx = x \circ S$.) Directly from the last equation

$$\overline{x} \circ \overline{S}(\mathcal{F}) = \mathcal{F}\text{-}\lim Tx \tag{6}$$

we get using linearity of $\mathcal{F}$-lim that

$$\overline{S}_n(\overline{x})(\mathcal{F}) = \frac{\overline{x} + \overline{x} \circ \overline{S} + \cdots + \overline{x} \circ \overline{S}^{n-1}}{n}(\mathcal{F}) = \mathcal{F}\text{-}\lim T_n(x). \tag{7}$$

Note that in the derivation of (7) we did not use the form of the map S. (The same holds for any map $S\colon \mathbb{N} \to \mathbb{N}$ and corresponging $T\colon x \mapsto x \circ S$.)

In this specific case it would possible to show that $\overline{S}(\mathcal{F}) = \{A - 1; A \in \mathcal{F}\}$, where $A - 1 = \{n \in \mathbb{N}; n + 1 \in A\}$.

It is useful to notice that (6) means that

$$\overline{S}\mathcal{F}\text{-}\lim x = \mathcal{F}\text{-}\lim Tx. \tag{8}$$

4.1.1 Categorical viewpoint

This part can be omitted when reading this text, but for a reader with background in category theory, this might be a useful insight.

Similarly as in (4), for any continuous map $f\colon X \to Y$ we can obtain a map $\beta f\colon \beta X \to \beta Y$.

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & \beta X \arrow[d, "\beta f"] \\ Y \arrow[r, "i_Y"'] & \beta Y \end{tikzcd}$$

It can be shown that $\beta(g \circ f) = \beta g \circ \beta f$, which means that β is a functor from the category **CReg** of completely regular spaces to the category $\mathbf{CompT_2}$ of compact Hausdorff spaces. In particular, this implies (5). ($\overline{x \circ S} = \overline{x} \circ \overline{S}$ is just a different notation for $\beta(x \circ S) = \beta x \circ \beta S$.)

The condition (1) from the definition of Stone-Čech compactification means, that the functor β is left adjoint to the embedding functor $E\colon \mathbf{CompT_2} \to \mathbf{CReg}$. (It says that $C(\beta X, Y) \cong C(X, EY)$.) In the other words, $\mathbf{CompT_2}$ is a reflective subcategory of **CReg** and the functor β is the corresponding reflector.

4.2 Expressing $M(x)$

Let us isolate one result obtained in the proof of [J2, Theorem 3] which might be of independent interest.

Proposition 4. *For every $x \in \ell_\infty$ there exists a free ultrafilter $\mathcal{G} \in \beta\mathbb{N}^*$ such that*

$$\lim_{n\to\infty} \mathcal{G}\text{-}\lim T_n(x) = M(x) = \sup_{\psi\in\mathcal{BL}} \psi(x),$$

where $\mathcal{BL}$ denotes the set of all Banach limits.

Proof. Let us fix some $x \in \ell_\infty$.

Since the set $\mathcal{BL}$ is compact in weak* topology and the function x^* is continuous with respect to this topology, there exists a Banach limit ψ_0 such that $x^*(\psi_0) = \sup_{\psi\in\mathcal{BL}} x^*(\psi)$, i.e.

$$\psi_0(x) = \sup_{\psi\in\mathcal{BL}} \psi(x) = M(x).$$

As described in Subsection 1.2.2, we can identify ℓ_∞^* and $C^*(\beta\mathbb{N})$. Thus by Riesz representation theorem the functional ψ_0 can be rewritten as

$$\psi_0(x) = \int_{\mathcal{F}\in\beta\mathbb{N}} \overline{x}(\mathcal{F})\, \mathrm{d}\mu = \int_{\mathcal{F}\in\beta\mathbb{N}} \mathcal{F}\text{-}\lim x\, \mathrm{d}\mu$$

for some regular Borel measure μ on $\beta\mathbb{N}$.

We now verify that the assumptions of Birkhoff's ergodic theorem are fulfilled for the map $\overline{S}\colon \beta\mathbb{N} \to \beta\mathbb{N}$. function $\overline{x}\colon \beta\mathbb{N} \to \mathbb{R}$ and the measure μ. We also show some properties of μ that will be needed in the proof.

Integrability. We have $|\overline{x}(\mathcal{F})| = |\mathcal{F}\text{-lim}\, x| = \mathcal{F}\text{-lim}|x|$, thus

$$\int_{\mathcal{F}\in\beta\mathbb{N}} |\overline{x}(\mathcal{F})| \,\mathrm{d}\mu = \psi_0(|x|) < +\infty$$

and $\overline{x} \in L_1(\mu)$.

Invariance. We should verify that the measure μ is $\overline{S}$-invariant, i.e. $\mu(B) = \mu(\overline{S}^{-1}B)$ for every Borel set. In fact, it suffices to verify the $\overline{S}$-invariance for set from the base $\{U_A; A \subseteq \mathbb{N}\}$ (see [EW, Theorem A.8]).[1]

We have

$$\mu(U_A) = \psi_0(\chi_A)$$

since

$$\mathcal{F}\text{-lim}\,\chi_A = \begin{cases} 1 & A \in \mathcal{F} \Leftrightarrow \mathcal{F} \in U_A, \\ 0 & A \notin \mathcal{F} \Leftrightarrow \mathcal{F} \notin U_A. \end{cases}$$

i.e. $\mathcal{F}\text{-lim}\,\chi_A = \chi_{U_A}$.

From

$$\overline{S}\mathcal{F} \in U_A \Leftrightarrow \overline{\mathrm{S}}\mathcal{F}\text{-lim}\,\chi_A = 1 \overset{(8)}{\Leftrightarrow} \mathcal{F}\text{-lim}\, T\chi_A = \mathcal{F}\text{-lim}\,\chi_{S^{-1}A} = 1 \Leftrightarrow S^{-1}A \in \mathcal{F}$$

we get $\overline{S}^{-1}U_A = U_{S^{-1}A}$ and

$$\mu(\overline{S}^{-1}U_A) = \mu(U_{S^{-1}A}) = \psi_0(\chi_{S^{-1}A}) = \psi_0(T\chi_A) = \psi_0(\chi_A) = \mu(U_A).$$

μ vanishes on $\mathbb{N}$. In the proof, we will also use the fact that $\mu(\mathbb{N}) = 0$. It suffices to note that $\mu(\{n\}) = \psi_0(\chi_{\{n\}}) = 0$ and then use the σ-additivity of measure.

From ergodic theorem we get that $\overline{S}_n(\overline{x})$ converges almost everywhere to some function $X\colon \beta\mathbb{N} \to \mathbb{R}$, i.e., there exists $\Delta \subseteq \beta\mathbb{N}^*$ such that $\mu(\Delta) = 1$ and

$$\lim_{n\to\infty} \overline{S}_n(\overline{x})(\mathcal{G}) = \lim_{n\to\infty} \mathcal{G}\text{-lim}\, T_n x = X(\mathcal{G})$$

for each $\mathcal{G} \in \Delta$. (Since $\mu(\mathbb{N}) = 0$, we can assume that $\Delta \subseteq \beta\mathbb{N}^*$.) We can assume that $X(\mathcal{G}) = 0$ for $\mathcal{G} \notin \Delta$. (We are changing X only on a set of measure zero.)

For every $\mathcal{G} \in \Delta$ we have[2]

$$\mathcal{G}\text{-lim}\, T_n(x) \le \limsup T_n(x),$$
$$X(\mathcal{G}) = \lim_{n\to\infty} \mathcal{G}\text{-lim}\, T_n(x) \le \lim_{n\to\infty} \limsup T_n(x) = M(x) = \psi_0(x).$$

From ergodic theorem we also have

$$\int_{\mathcal{G}\in\beta\mathbb{N}} X(\mathcal{G}) \,\mathrm{d}\mu = \int_{\mathcal{G}\in\beta\mathbb{N}} \overline{x}(\mathcal{G}) = \psi_0(x).$$

[1] To use [EW, Theorem A.8], we should verify that $\mathcal{S} = \{U_A; A \subseteq \mathbb{N}\}$ is a semi-algebra. We have $\emptyset = U_{\mathbb{N}} \in \mathcal{S}$. It is closed under finite intersections: $U_{A\cap B} = U_A \cap U_B$. Complement of set from $\mathcal{S}$ is from $\mathcal{S}$ since $U_{\mathbb{N}\setminus A} = \beta\mathbb{N} \setminus U_A$.

[2] A different possibility to show $X(\mathcal{G}) \le M(x)$ would be using the fact that $\mathcal{F}\text{-lim}_n\, \mathcal{G}\text{-lim}\, T_n(x)$ is a Banach limit for every free ultrafilter $\mathcal{F}$. In this approach we do not need to now the explicit form of $M(x)$ beforehand.

If we combine this fact with the the inequality $X(\mathcal{G}) \leq \psi_0(x)$ we get that the equality
$$X(\mathcal{G}) = \psi_0(x)$$
holds for μ-almost all $\mathcal{G}$'s in Δ. For every such $\mathcal{G}$ we get
$$\lim_{n\to\infty} \mathcal{G}\text{-}\lim T_n(x) = X(\mathcal{G}) = \psi_0(x) = M(x).$$
□

4.3 Extreme Banach limits

Lemma 2. *For any free ultrafilters $\mathcal{F}$, $\mathcal{G}$ the functional $f\colon \ell_\infty \to \mathbb{R}$ defined by*
$$f(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim T_n(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim_k \frac{x_k + \cdots + x_{k+n-1}}{n}$$
is a Banach limit.

Proof. Linearity and positivity are obvious.

If can be shown that if (x_n) converges to l, then $\frac{x_k+\cdots+x_{k+n-1}}{n}$ converges to l uniformly in k. This implies that also $\mathcal{G}\text{-}\lim T_n(x)$ converges to l and that $f(x) = l$. Thus f extends limit.

The functional f is also shift-invariant since
$$f(Tx - x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim_k \frac{x_{k+n} - x_k}{n} = 0.$$
(We are using again the fact that the sequence x is bounded: we have $\frac{x_{k+n}-x_k}{n} \leq \frac{\|x\|}{n}$ for each k, which implies $\mathcal{G}\text{-}\lim_k \frac{x_{k+n}-x_k}{n} \leq \frac{\|x\|}{n}$.) □

Combining the above results we get

Theorem 6 ([J2, Theorem 3])**.** *Let Q denote the set all linear functionals of the form*
$$f(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim T_n(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim_k \frac{x_k + \cdots + x_{k+n-1}}{n},$$
where $\mathcal{F}$ and $\mathcal{G}$ are free ultrafilters on $\mathbb{N}$. Let $\mathcal{BL}$ denote the set of all Banach limits. Then $Q \subseteq \mathcal{BL}$ and $\mathcal{BL} = \overline{\text{co}}(Q)$.

Proof. We have $Q \subseteq \mathcal{BL}$ from Lemma 2. By Lemma 1, the set $\mathcal{BL}$ is convex compact subset of ℓ_∞^* (with the weak*-topology). Now by Proposition 2 it suffices to show
$$(\forall x \in \ell_\infty) \sup_{f\in Q} f(x) = \sup_{f\in\mathcal{BL}} f(x) = M(x).$$
This follows from Proposition 4. □

Let us note that some other sets fulfilling $\overline{\text{co}}(A) = \mathcal{BL}$ or even containing all extreme points of $\mathcal{BL}$ are described in [J2] and [Š]. (Several such sets can be obtained relatively easily from various expressions of $M(x)$.)

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