Meyer Jerison: The set of all generalized limits of bounded sequences

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Overview

We present some results from [J2].

- Proof of existence of Banach limits (using Hahn-Banach theorem, using ultrafilters).
- Some results on the extreme points of the set of all Banach limits.

These slides and more detailed notes are available at: `http://thales.doa.fmph.uniba.sk/sleziak/papers/semtrf.html`

Filters

Definition

A filter on a set M is a non-empty family $\mathcal{F} \subseteq \mathcal{P}(M)$ such that:

i. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

ii. $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$;

iii. $\emptyset \notin \mathcal{F}$.

A filter is called *free* if $\bigcap \mathcal{F} = \emptyset$.

Example: Fréchet filter = cofinite subsets of $\mathbb{N}$

Ultrafilters

Definition

A filter $\mathcal{F}$ on M is an *ultrafilter* if for any $A \subseteq M$

$$A \in \mathcal{F} \qquad \vee \qquad M \setminus A \in \mathcal{F}.$$

Ultrafilters are precisely maximal filters (with respect to inclusion).
AC $\Rightarrow$ Every system of subsets of M, that has finite intersection property, is contained in an ultrafilter. $\Rightarrow$ Free ultrafilters exist.
$\mathcal{F}_m = \{A \subseteq M; m \in A\}$ = principal ultrafilter (not free)

Limit along a filter

Definition

If (x_n) is a sequence of elements of a topological space X and $\mathcal{F}$ is a filter on $\mathbb{N}$, then we say that (x_n) is $\mathcal{F}$-convergent to $l \in X$ if for every neighborhood U of l the set

$$x^{-1}(U) = \{n \in \mathbb{N}; x_n \in U\}$$

belongs to $\mathcal{F}$.
Notation: $\mathcal{F}\text{-}\lim x_n = l$.

Usual limit = $\mathcal{F}$-limit for Fréchet filter

Limit along a filter

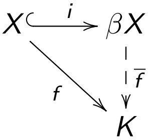
Properties:

- uniqueness in Hausdorff spaces;
- additive, multiplicative;
- If $\mathcal{F}$ is ultrafilter and X is compact, then $\mathcal{F}$-limit exists.
- $\mathcal{F}$-limits of a sequence (x_n) for free (ultra)filters $\mathcal{F}$ are precisely all cluster points of this sequence.
- $\mathcal{F}$-lim extends the usual limit, if $\mathcal{F}$ is free.

Stone-Čech compactification

Definition

Let X be a topological space. Then a compact Hausdorff space βX is *Stone-Čech compactification* of X if there exists embedding $i\colon X \hookrightarrow \beta X$ such that for every continuous map $f\colon X \to K$ to a compact Hausdorff space K there exists a unique continuous extension $\overline{f}\colon \beta X \to K$ fulfilling $\overline{f} \circ i = f$.



$\beta\mathbb{N}$

- Uderlying set: $\mathrm{Uft}(\mathbb{N})$ = all ultrafilters on $\mathbb{N}$
- topology given by the base $\{U_A; A \subseteq \mathbb{N}\}$, where

$$U_A = \{\mathcal{F} \in \mathrm{Uft}(\mathbb{N}); A \in \mathcal{F}\}$$
$$\mathcal{F} \in U_A \qquad \Leftrightarrow \qquad A \in \mathcal{F}$$

- Embedding: $i\colon n \mapsto \mathcal{F}_n = \{B \subseteq \mathbb{N}; n \in B\}$
- Extension of a bounded sequence $x\colon \mathbb{N} \to \mathbb{R}$:

$$\overline{x}(\mathcal{F}) = \mathcal{F}\text{-}\lim x,$$

$C(\beta\mathbb{N})$ and $\ell_\infty(\mathbb{N})$

Proposition

Banach spaces ℓ_∞ and $C(\beta\mathbb{N})$ are isometrically isomorphic, i.e., there exists a norm-preserving linear isomorphism between them.

$\varphi\colon \ell_\infty \to C(\beta\mathbb{N})$ (extension)

$$\varphi(x) = \overline{x}$$

$\psi\colon C(\beta\mathbb{N}) \to \ell_\infty$ (restriction)

$$\psi(f) = f \circ i$$

Weak*-topology

For every $x \in X$ we have a linear map $x^*\colon X^* \to \mathbb{R}$

$$x^*(f) = f(x).$$

Definition

The *weak*-topology* is the weakest topology on X^* such that all maps x^* are continuous with respect to this topology.

It is the topology induced by the product topology on $\mathbb{R}^X$.

$f_d \to f$ in weak* topology $\Leftrightarrow$ $f_d(x) \to f(x)$ for each $x \in X$

Banach-Alaoglu theorem

Theorem (Banach-Alaoglu)

Let $B = \{f \in X^*; \|f\| \leq 1\}$ be the unit ball of X^*. The set B is compact in the weak*-topology on X^*.

Extreme points

Definition

Let C be a subset of a topological vector space E. Let $e \in C$. The point e is an *extreme point* of the set C if for $x_{1,2} \in C$

$$\frac{x_1 + x_2}{2} = e \qquad \Rightarrow \qquad x_1 = x_2 = e$$

holds. In the other words: The point e cannot be expressed as a non-trivial convex combination of points from C.

Extreme points

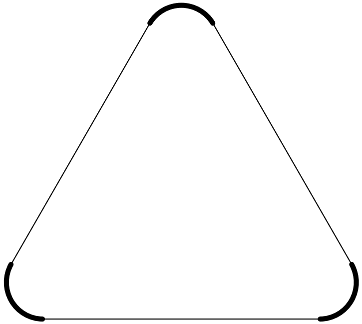


Figure: Extreme points of a set (taken from [WIK])

Krein-Milman theorem

Theorem ([J1, Theorem1])

Let E be a locally convex TVS and C be a compact convex subset of E. Let $S \subseteq C$. The following assertions are equivalent:

i. *For every linear continuous function $f\colon E \to \mathbb{R}$ the equality*

$$\sup_{x \in S} f(x) = \sup_{x \in C} f(x)$$

holds;

ii. $C = \overline{\operatorname{co}}(S)$, *i.e. C is the closed convex hull S;*

iii. *the closure $\overline{S}$ of the set S contains all extreme points of C.*

Krein-Milman theorem

Corollary (Krein-Milman)

Let E be a locally convex TVS and C be a compact convex subset of E. Then C is the closed convex hull of the set of all extreme points of C.

$$C = \overline{\text{co}}(\text{Ext}(C))$$

Reformulation for X^*

Linear functionals on E are precisely the maps x^* for $x \in X$ (see [FHH$^+$, Proposition 3.22])

Proposition

Let X be a linear normed space and C be a subset of X^* which is convex and compact in the weak-topology. Let $S \subseteq C$. The following conditions are equivalent:*

i.
$$\sup_{\varphi \in S} \varphi(x) = \sup_{\varphi \in C} \varphi(x) \tag{1}$$
holds for each $x \in X$;

ii. *$C = \overline{\text{co}}(S)$, i.e. C is the closed convex hull S;*

iii. *the closure $\overline{S}$ of the set S contains all extreme points of C.*

Measure preserving system

Definition

A quadruple $(X, \mathcal{B}, \mu, T)$ is a measure preserving system if $\mathcal{B}$ is a σ-algebra on X, μ is a measure on $\mathcal{B}$ and map $T\colon X \to X$ is measurable and fulfills the condition

$$\mu(T^{-1}A) = \mu(A)$$

for each $A \in \mathcal{B}$.

Birkhoff's ergodic theorem

Also known as: pointwise ergodic theorem, individual ergodic theorem.

Theorem ([EW, Theorem 2.30])

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. If $f \in L_1(\mu)$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = f^*(x)$$

converges almost everywhere and in $L_1(\mu)$ to a T-invariant function $f^* \in L_1(\mu)$ and

$$\int f^* \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

Averages

For $T\colon X \to X$ and $f\colon X \to \mathbb{R}$ we will use the notation $T_n(f) = \frac{f+fT+\cdots+fT^{n-1}}{n}$. I.e., $T_n(f)\colon X \to \mathbb{R}$ is the function given by

$$T_n(f)(x) = \frac{f(x) + f(Tx) + \cdots + f(T^{n-1}x)}{n} \tag{2}$$

for any $x \in X$.

Hence the Birkhoff's ergodic theorem says that for every measure preserving transformation the sequences $T_n(f)$ converges pointwise to some f^* and that f^* and f have the same integral.

Regular Borel measure

Regular Borel measure is a measure μ on a topological space with the following properties:

- μ is defined on the σ-algebra $\mathcal{B}$ of all Borel sets;
- $\mu(K) < \infty$ for every compact subset;
- $\mu(B) = \sup\{\mu(K); K \subseteq B; K \text{ is compact}\}$ for every $B \in \mathcal{B}$;
- $\mu(B) = \inf\{\mu(U); U \supseteq B; U \text{ is open}\}$ for every $B \in \mathcal{B}$ such that $\mu(B) < \infty$. (If we assume the validity of the last condition for open sets, we get an equivalent definition.)

Riesz Representation Theorem

Theorem (Riesz Representation Theorem, [AB, Theorem 38.3])

Let X be a compact Hausdorff space. For every positive linear functional F on $C(X)$, there exists a unique regular Borel measure μ on X such that

$$F(f) = \int_X f \, \mathrm{d}\mu$$

holds for every $f \in C(X)$.

Extensions of limit

Limit: $\lim\colon c \to \mathbb{R}$ linear functional.
Looking for: extensions $f\colon \ell_\infty \to \mathbb{R}$ which are linear and have similar properties to the usual limit.
Example: $\mathcal{F}\text{-}\lim\colon \ell_\infty \to \mathbb{R}$ (for any free ultrafilter $\mathcal{F}$)
Multiplicative:

$$f(x.y) = \mathcal{F}\text{-}\lim x_n.y_n = \mathcal{F}\text{-}\lim x_n.\,\mathcal{F}\text{-}\lim y_n = f(x).f(y).$$

Shift-invariance

Let us define shift-operator $T\colon \ell_\infty \to \ell_\infty$ by

$$T\colon (x_n) \mapsto (x_{n+1}).$$

A functional $f \in \ell_\infty^*$ is said to be *shift-invariant* if

$$f(Tx) = f(x).$$

A linear functional $f \in \ell_\infty^*$, which extends limit, cannot be simultaneously multiplicative and shift-invariant:
$x = (1, 0, 1, 0, \dots)$

- shift-invariant: $f(x) + f(Tx) = f(\bar{1}) = 1 \Rightarrow$ $f(x) = f(Tx) = \frac{1}{2}$
- multiplicative: $f(x)^2 = f(x) \Rightarrow f(x) \in \{0, 1\}$

Banach limit

Definition

A linear functional $f\colon \ell_\infty \to \mathbb{R}$ is called *Banach limit,* if it is positive, shift-invariant and extends limit, i.e.,

i. $x \geq 0 \Rightarrow f(x) \geq 0$;

ii. $(\forall x \in \ell_\infty) f(Tx) = f(x)$;

iii. if x is convergent then $f(x) = \lim x$.

$$\liminf x_n \leq f(x) \leq \limsup x_n$$

$f \in \ell_\infty^*$ and $\|f\| = 1$

Sequence $T_n(x)$

For any bounded sequence x we define $T_n(x) = \frac{x+Tx+\cdots+T^{n-1}x}{n}$.
I.e., $T_n(x)$ is the sequence $\left(\frac{x_k+x_{k+1}+\cdots+x_{k+n-1}}{n}\right)_{k=1}^{\infty}$

$$M(x) = \lim_{n\to\infty} \limsup T_n(x)$$
$$m(x) = \lim_{n\to\infty} \liminf T_n(x)$$

Sequence $T_n(x)$

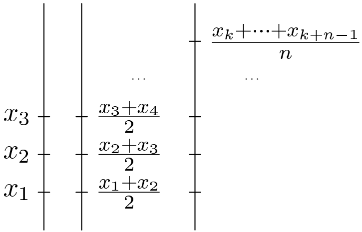


Figure: The sequences $T_n(x)$

Extreme values of Banach limits

Proposition

For a given bounded sequence x all possible values of Banach limits are the values from the interval $\langle m(x), M(x)\rangle$.

Proof: Using Hahn-Banach theorem for the functional lim majorized by $M(x)$.
Another proof: Using ultralimits:

$$f(x) = \mathcal{F}\text{-}\lim \frac{x_{p_n} + \cdots + x_{p_n+n-1}}{n}$$

Hahn-Banach theorem

Theorem (Hahn-Banach theorem)

Let X be a vector space and let $p\colon X \to \mathbb{R}$ be any sublinear function. Let M be a vector subspace of X and let $f : M \to \mathbb{R}$ be a linear functional dominated by p on M. Then there is a linear extension $\hat{f}$ of f to X that is dominated by p on X.

Hahn-Banach theorem

Theorem (Hahn-Banach theorem)

Moreover, for given given $v \in X$, there exists an extension such that the value $\hat{f}(v) = c$ if and only if

$$\sup_{x \in M} [f(x) - p(x - v)] \leq c \leq \inf_{y \in M} [p(y + v) - f(y)].$$

In case the p and f have the additional property that

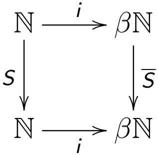
$$(\forall x \in X)(\forall y \in M) p(x + y) = p(x) + f(y)$$

then the above interval can be simplified to

$$-p(-v) \leq c \leq p(v).$$

Shift on $\beta\mathbb{N}$

$S\colon \mathbb{N} \to \mathbb{N}$ given by $S(n) = n + 1$ induces $\overline{S}\colon \beta\mathbb{N} \to \beta\mathbb{N}$:



Shift on $\beta\mathbb{N}$

$$\overline{x \circ S} = \overline{x} \circ \overline{S}$$

$$\overline{x} \circ \overline{S}(\mathcal{F}) = \overline{x \circ S}(\mathcal{F}) = \overline{Tx}(\mathcal{F}) = \mathcal{F}\text{-lim } Tx$$

$$\overline{x} \circ \overline{S}(\mathcal{F}) = \mathcal{F}\text{-lim } Tx$$

$$\overline{S}\mathcal{F}\text{-lim}\, x = \mathcal{F}\text{-lim } Tx$$

$$\overline{S}_n(\overline{x})(\mathcal{F}) = \frac{\overline{x} + \overline{x} \circ \overline{S} + \cdots + \overline{x} \circ \overline{S}^{n-1}}{n}(\mathcal{F}) = \mathcal{F}\text{-lim } T_n(x)$$

Expressing $M(x)$

Proposition

For every $x \in \ell_\infty$ there exists a free ultrafilter $\mathcal{G} \in \beta\mathbb{N}^*$ such that

$$\lim_{n\to\infty} \mathcal{G}\text{-}\lim T_n(x) = M(x) = \sup_{\psi\in\mathcal{BL}} \psi(x),$$

where $\mathcal{BL}$ denotes the set of all Banach limits.

Expressing $M(x)$

Sketch of the proof:
Compactness (Banach-Alaoglu):

$$\psi_0(x) = \sup_{\psi \in \mathcal{BL}} \psi(x) = M(x)$$

ℓ_∞^* as $C^*(\beta\mathbb{N})$ + Riesz representation Theorem:

$$\psi_0(x) = \int_{\mathcal{F}\in\beta\mathbb{N}} \overline{x}(\mathcal{F})\,\mathrm{d}\mu = \int_{\mathcal{F}\in\beta\mathbb{N}} \mathcal{F}\text{-}\lim x\,\mathrm{d}\mu$$

Expressing $M(x)$

$\mu(\mathbb{N}) = 0$ and μ fulfills the assumptions of ergodic theorem $\Rightarrow$ there exists a pointwise limit X:

$$\lim_{n\to\infty} \overline{S}_n(\overline{x})(\mathcal{G}) = \lim_{n\to\infty} \mathcal{G}\text{-}\lim T_n x = X(\mathcal{G})$$

$$\int_{\mathcal{G}\in\beta\mathbb{N}} X(\mathcal{G})\, \mathrm{d}\mu = \int_{\mathcal{G}\in\beta\mathbb{N}} \overline{x}(\mathcal{G}) = \psi_0(x)$$

$$X(\mathcal{G}) \leq \psi_0(x) = M(x)$$

This implies that $X(\mathcal{G}) = \psi_0(x)$ holds for μ-almost all $\mathcal{G}$'s. For every such $\mathcal{G}$ we get

$$\lim_{n\to\infty} \mathcal{G}\text{-}\lim T_n(x) = X(\mathcal{G}) = \psi_0(x) = M(x).$$

Extreme Banach limits

Lemma

For any free ultrafilters $\mathcal{F}$, $\mathcal{G}$ the functional $f\colon \ell_\infty \to \mathbb{R}$ defined by

$$f(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim T_n(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim_k \frac{x_k + \cdots + x_{k+n-1}}{n}$$

is a Banach limit.

Extreme Banach limits

Theorem ([J2, Theorem 3])

Let Q denote the set all linear functionals of the form

$$f(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim T_n(x) = \mathcal{F}\text{-}\lim_n \mathcal{G}\text{-}\lim_k \frac{x_k + \cdots + x_{k+n-1}}{n},$$

where $\mathcal{F}$ and $\mathcal{G}$ are free ultrafilters on $\mathbb{N}$. Let $\mathcal{BL}$ denote the set of all Banach limits. Then $Q \subseteq \mathcal{BL}$ and $\mathcal{BL} = \overline{\text{co}}(Q)$.

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