1 E. van Douwen: Finitely additive measures on $\mathbb{N}$

1.1 Introduction

$\mu\colon \mathcal{P}(S) \to \langle 0,\infty\rangle$ is called a *measure* on S if $\mu(S)=1$ and $\mu(X\cup Y) = \mu(X)+\mu(Y)$ for every 2 disjoint subsets of S.

A measure μ on S is said to be *shift-invariant* if $\mu(1+X)=\mu(X)$.

$r*K=\{r*K; k\in K\}$ for $r\in\langle 1,\infty)$, $K\subseteq\mathbb{N}$ and $*\in\{+,\cdot\}$
$[X]=\{[x] : x\in X\}$ for $X\subseteq\langle 1,\infty)$
If $K\subseteq\mathbb{N}$ (the unique strictly increasing surjection $c_K\colon\mathbb{N}\to K$ = counting function) and if f is any function from $\mathbb{N}$ to $\mathbb{N}$, we put
$f\circ K=\{n\in\mathbb{N} : (\exists k\in K)[n \text{ is the } f(k)\text{th member of } \mathbb{N}]\}$ (=range of $f\circ c_K$)
$K\circ f=\{k\in K : (\exists n\in\mathbb{N})[k \text{ is the } f(n)\text{th member of } K]\}$ (=range of $c_K\circ f$)
Finally, for $K\subseteq\mathbb{N}$ and $m\in\mathbb{N}$ we put
$K\cdot m=K\circ f$, where $f\colon\mathbb{N}\to\mathbb{N}$ is defined by $f(n)=mn$.

Let $f\colon\mathbb{N}\to\mathbb{N}$ be an injection. We denote

$$\lambda(f)=\lim_{n\to\infty}\frac{n}{f(n)}$$

if the limit exists. The set of all injections for which $\lambda(f)$ exist will by denote by $\mathbb{L}$.

For a measure μ on $\mathbb{N}$ and for $A\subseteq\langle 1,\infty)$ we define the following properties of μ.

(1) μ is *diffuse* if μ vanishes on finite sets;

(2) μ is *A-scale-invariant* if $\mu([a\cdot K])=a^{-1}.\mu(K)$ for each $a\in A$ and each $K\subset\mathbb{N}$.

(3) μ is *A-stretchable* if $\mu([b+a\cdot K])=a^{-1}.\mu(K)$ for each $a\in A$, each $b\in\langle 0,\infty)$ and each $K\subset\mathbb{N}$.

(4) μ is *stretchable* if $\mu(f\circ K)=\lambda(f).\mu(K)$ for each $f\in\mathbb{L}$ and for each $K\subseteq\mathbb{N}$.

(5) μ is *thinnable* if $\mu(K\circ f)=\lambda(f).\mu(K)$ for each $f\in\mathbb{L}$ and for each $K\subseteq\mathbb{N}$.

(6) μ *extends density* if $\mu(K)=d(K)$ for each $K\subseteq\mathbb{N}$such that $d(K)$ exists.

(7) μ *extends density* r if $\mu(K)=r$ provided $d(K)=r$.

Proposition. *Let $f\colon\mathbb{N}\to\mathbb{N}$ be an injection, and let $K=f\circ\mathbb{N}=\mathbb{N}\circ f$.*

(i) If $\lambda(f)$ exists then $d(K)=\lambda(f)$.

(ii) If f is strictly increasing and if $d(K)$ exists then $f \in \mathbb{L}$ and $\lambda(f) = d(K)$.

Remark. (1.2) Stretchable and thinnable measures extend density. (The converse is false by Example 5.6.)

Theorem. *(1.3) $\mathbb{N}$ has a measure which is both stretchable and thinnable.*

Theorem. *(1.4) There are a $\langle 1, \infty)$-stretchable measure μ on $\mathbb{N}$ and a $K \subseteq \mathbb{N}$ such that $\mu(K) = 1$ but $d(K) = 0$.*

Theorem. *(1.5) $\mathbb{N}$ has a thinnable measure which is no stretchable.*

Question. (1.6) Is each stretchable measure thinnable?

Theorem. *(1.7) There are a $\mathbb{Q}^{\geq 1}$-stretchable measure μ on $\mathbb{N}$ and a $K \in \mathcal{P}(\mathbb{N})$ so that $\mu(K) = 1$ but $\mu([r \cdot K]) = 0$ for each irrational $r \in \langle 1, \infty)$.*

Fact. (1.8) Each $\mathbb{N}$-scale-invariant shift-invariant measure is $\mathbb{Q}^{\geq 1}$-stretchable.

Question. (1.9) Is there a $\langle 1, \infty)$-scale-invariant measure that is not shift-invariant?

Is there an $\mathbb{N}$-scale-invariant measure that is not $\{r\}$-scale-invariant for each noninteger $r \in \langle 1, \infty)$.

Theorem. *(1.10) For each $q \in \mathbb{Q}^{\geq 1} \setminus \mathbb{N}$ there are an $\mathbb{N}$-scale-invariant measure μ on $\mathbb{N}$ and a $K \subseteq \mathbb{N}$ such that $\mu(K) = 0$ but $\mu([q \cdot K]) \neq 0$ (and in fact $\mu([q \cdot K]) = n^{-1}$ if $q = n/m$ for relatively prime $n, m \in \mathbb{N}$), hence μ is not shift-invariant.*

Theorem. *(1.11) The following conditions on a measure μ on $\mathbb{N}$ are equivalent:*

(i) If $f(n) = [q+rn]$ for $n \in \mathbb{N}$, then $\mu(K \circ f) = r^{-1}.\mu(K)$, for each $r \in \langle 1, \infty)$;

(ii) $\mu(K \cdot 2) = \frac{1}{2}\mu(K)$ for each $K \subseteq \mathbb{N}$;

(iii) there is $m \in \mathbb{N} \setminus \{1\}$ such that $\mu(K \cdot m) = m^{-1}\mu(K)$ for each infinite $K \subseteq \mathbb{N}$;

(iv) μ is shift-invariant.

Theorem. *(1.12) The following conditions on a measure μ on $\mathbb{N}$ are*

(i) μ extends density;

(ii) there is $r \in (0, 1)$ such that μ extends density r;

(iii) there is $k \in \mathbb{N} \setminus \{1\}$ such that μ extends density k^{-1};

(iv) $\mu(f \circ K) = \mu(K)$ for each injection $f \in \mathbb{L}$ such that $\lambda(f) = 1$ and each $K \subseteq \mathbb{N}$;

(v) $\mu(f \circ K) = \mu(K)$ for each permutation $f \in \mathbb{L}$ such that $\lambda(f) = 1$ and each $K \subseteq \mathbb{N}$.

Theorem. *(1.13) The following conditions on a measure μ on $\mathbb{N}$ are*

(i) μ *extends density 0;*

(ii) μ *extends density 1;*

(iii) $\mu(K \circ f) = \mu(K)$ *for each injection* $f \in \mathbb{L}$ *such that* $\lambda(f) = 1$ *and each* $K \subset \mathbb{N}$*;*

(iv) $\mu(K \circ f) = 0$ *for each injection* $f \in \mathbb{L}$ *such that* $\lambda(f) = 0$ *and each* $K \subset \mathbb{N}$*;*

(v) $\mu(f \circ K) = 0$ *for each injection* $f \in \mathbb{L}$ *such that* $\lambda(f) = 0$ *and each* $K \subset \mathbb{N}$*.*

Theorem. *(1.14) Let* f *be a function from* $\mathbb{N}$ *to* $\langle 0, 1\rangle$*. There is a diffuse measure* μ *on* $\mathbb{N}$ *such that* $\mu(m \cdot K) = f(m) \cdot \mu(K)$ *for all* $m \in \mathbb{N}$ *and* $K \subseteq \mathbb{N}$ *if (and, trivially, only if)* $f(1) = 1$ *and* $f(kl) = f(k).f(l)$ *for all* $k, l \in \mathbb{N}$*. (In fact* μ *can be chosen to extend density 0.)*

Question. (1.15) What are the extreme points in the (closed convex) set of stretchable measures? Or the set of elastic measures (of Section 5)? (Those are the nicest measures I can currently think of.)

1.2 Special notation

We frequently denote the function f with domain D whose value at $d \in D$ is $f(d)$ by $\langle f_{(d)}\rangle_{d\in D}$.

1.3 Means

$\mathcal{B}(S)$= vector space of all bounded functions $f\colon S \to \mathbb{R}$, supremum norm

A function $\varphi\colon \mathcal{B}(S) \to \mathbb{R}$ will be called a *mean* on S if it satisfies the following two equivalent conditions:

1. $\varphi(1) = 1$ and φ is positive,
2. $\inf_{s\in S} x_s \le \varphi(x) \le \sup_{s\in S} x_s$

diffuse mean $\Leftrightarrow$ extends limits $\Leftrightarrow$ $\liminf \le \varphi \le \limsup$ $\Leftrightarrow$ $\varphi(x) \ge 0$ whenever $x_s \ge 0$ for all but finitely many $s \in S$.

If φ is a diffuse mean, we write sometimes $\varphi - \lim x_n$ instead of $\varphi((x_n))$.

Finally, if S is a multiplicatively written semigroup then a mean φ on S will be called *left-invariant* if $\varphi(\langle x_{ts}\rangle_{s\in S}) = \varphi(x)$ for all $t \in S$ and $\mathcal{B}(S)$.

invariant mean = left and right invariant

There is a natural one-to-one correspondence between means and measures on a set S. Given a mean φ one can define a measure μ on S by

$$\mu(X) = \varphi(\chi_X)$$

and given a measure μ one can find a unique mean on S that satisfies $\mu(X) = \varphi(\chi_X)$ by imitating constructions of the Lebesgue integral, see e.g. [vN, footnote 37].

Proposition. *(3.3) If φ and μ are as above, then φ is diffuse iff μ is diffuse.*

Proposition. *(3.5) Let μ be a measure on a multiplicatively written semigroup S and let φ be the associated mean. Then μ is left-invariant if and only if*

$$\mu(t^{-1}X) = \mu(X) \text{ for all } t \in S \text{ and } X \subseteq S,$$

where $t^{-1}X = \{s \in S : ts \in X\}$ (so $t^{-1}X = t^{-1} \cdot X$ if S is a group).

In proof of Theorem 3.7: Shift-invariant measure in invariant on $(\mathbb{N}, +)$.

1.4 Elastic measures

Elastic measure:
(1) μ is $\mathbb{N}$-scale invariant,
(2) $f \leq g \Rightarrow \mu(f \circ \mathbb{N}) \geq \mu(g \circ \mathbb{N})$ (for any $f, g \colon \mathbb{N} \to \mathbb{N}$)

Proposition. *(5.7) If a measure on $\mathbb{N}$ extends density then it is shift-invariant.*

1.5 On G-invariant measures

(Apendix 5A)
G is a group, S is a set. Action $\pi \colon G \times S \to S$.
G-invariant measure: $\mu(g * X) = \mu(X)$

1.6 A conjecture of Bumby and Ellentuck and question

$\mathcal{A}$ =measures on $\mathbb{N}$ such that $\varphi(x) \leq \sup n^{-1} \sum_{k=1}^{n} x_k$ (φ is the associated mean, L is Cesaro mean)
$\mathcal{A}'$=measures on $\mathbb{N}$ such that $\varphi(x) \leq \limsup n^{-1} \sum_{k=1}^{n} x_k$
$\mathcal{B} \Leftrightarrow \mu([r \cdot K]) \leq \mu(K)$ for $r \in [1, \infty)$.
It holds $\mathcal{A} \subsetneq \mathcal{B}$.
$\mathcal{A}'$ is precisely the set of diffuse measures in $\mathcal{A}$

Question. (7A.1) Let $\mathcal{C}$ be the set of all measures μ on $\mathbb{N}$ such that

$$\mu(K) \leq \limsup_{n \to \infty} \frac{K(n)}{n},$$

and let $\mathcal{D}$ =all density measures. So $\mathcal{A}' \subseteq \mathcal{C} \subseteq \mathcal{D}$. Is $\mathcal{A}' = \mathcal{C}$? Is $\mathcal{C} = \mathcal{D}$?

Asked again in van Mill: Open problems in van Douwen's papers
Is $\mathcal{C} \subseteq \mathcal{B}$? Is $\mathcal{D} \subseteq \mathcal{B}$?

References

[vN] J. von Neumann. Zur allgemeinen Theorie des Maes. *Fund. Math.*, 13:73–116, 1929.