Hamel basis and additive functions

[1]

Hamel basis

References: [Hei, Section 4.1], [Ku, Section 4.2, Chapter 11], [NS, Kapitola 4.7], [A, Section 6F] [2]

Existence of Hamel basis

Definition 1. Let V be a vector space over a field K. We say that B is a *Hamel basis* in V if B is linearly independent and every vector $v \in V$ can be obtained as a linear combination of vectors from B.

This is equivalent to the condition that every $x \in V$ can be written in precisely one way as
$$\sum_{i\in F} c_i x_i$$
where F si finite, $c_i \in K$ and $x_i \in B$ for each $i \in F$.

It is also easy to see that for any vector space W and any map $g\colon B \to W$ there exists exactly one linear map $f\colon V \to W$ such that $f|_B = g$.

Theorem 1. *Let V be a vector space over K. Let A be an linearly independent subset of V. Then there exist a Hamel basis B of V such that $A \subseteq B$. (Any linearly independent set is contained in a basis.)*

Proof. Zorn's lemma. □

Corollary 1. *Every vector space has a Hamel basis.*

Proof. For $V = \{0\}$ we have a basis $B = \emptyset$.

If $V \neq \{0\}$, we can take any non-zero element $x \in V$ and use Theorem 1 for $A = \{x\}$. □

In some cases we are able to write down a basis explicitly, for example in finitely-dimensional space or in the following example. However, the claim that a Hamel basis exists for each vector space over any field already implies AC (see [HR, Form 1A]).

Example 1. Let c_{00} be the space of all real sequences which have only finitely many non-zero terms. Then $\{e^{(i)}; i \in \mathbb{N}\}$, where the sequence $e^{(i)}$ is given by $e^{(i)}_n = \delta_{in}$, is a Hamel basis of this space.

[1] Available at `http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/pozn/tm/`

[2] See also: `thales.doa.fmph.uniba.sk/sleziak/texty/rozne/AC/cont.pdf`

Cardinality of Hamel basis

Proposition 1. *If B_1, B_2 are Hamel bases of a vector space V, then* $\operatorname{card} B_1 = \operatorname{card} B_2$*.*

Because of the above result, it makes sense to define *Hamel dimension* of a vector space V as the cardinality of any of its bases.

Hamel bases in linear normed spaces and Banach spaces

[3]

Cardinality. Recall that a subset A of a topological space X is called *meagre* in X if it is a countable union of nowhere-dense sets. Baire category theorem: If X is a complete metric space, then X is not meagre in X; i.e., X cannot be obtained as a countable union of nowhere-dense sets. (Similar claim is true for locally compact Hausdorff spaces.)

Theorem 2. *Let X be an infinite-dimensional Banach space.*

a) *If S is a subspace of X which has countable Hamel basis, then X is meagre in X.*
b) *Any Hamel basis of X is uncountable.*

The proof uses Baire category theorem and the fact that every finitely-dimensional subspace of a Banach space is closed (see [FHH$^+$, Proposition 1.36]). The same argument can be used to show analogous result for completely metrizable topological vector spaces (see [AB, Corollary 5.23]).

The above result can be in fact improved: It can be shown that cardinality of infinite-dimensional Banach space is at least $\mathfrak{c}$. We will give here a proof from [L].

We first recall a few fact about almost disjoint families (see [BŠ, §III.1], [B, Theorem 5.35], [JW, Theorems 17.17, 17.18]).

Definition 2. Let $\mathcal{A} = \{A_i; i \in I\}$ be a system of subsets of X. We say that $\mathcal{A}$ is an *almost disjoint family* or *AD family* on X, if $\operatorname{card} A_i = \operatorname{card} X$ for each $i \in I$ and the intersection $A_i \cap A_j$ is finite for each $i, j \in I$, $i \neq j$.

Lemma 1. *If X is an infinite countable set then there is an AD family on X of cardinality $\mathfrak{c}$.*

Proof. We will work with $X = \mathbb{Q}$. (The obtained AD family can be transferred to any infinite countable set.)

For every $r \in \mathbb{R}$ there is an injective sequence $f_r \colon \mathbb{N} \to \mathbb{Q}$ of rational numbers, which converges to r. Put $A_r = f_r[\mathbb{N}]$. It is easy to see that $\{A_r; r \in \mathbb{R}\}$ is an AD family. □

[3] I should mention that I've learned about some of these results (and their proofs) from discussions at `http://math.stackexchange.com`. See `http://math.stackexchange.com/questions/74101/`, `http://math.stackexchange.com/questions/33282/` and `http://math.stackexchange.com/questions/79184/`.

There are several other nice proofs of the existence of an AD family with the above properties. For instance, we could use nodes of an infinite binary tree. See, for example, [G].[4]

Theorem 3. *If X is an infinite-dimensional Banach space then Hamel dimension of X is at least $\mathfrak{c}$.*

Proof. We first construct inductively systems $\{x_i; i \in \mathbb{N}\} \subseteq X$ and $\{x_i^*; i \in \mathbb{N}\} \subseteq X^*$ such that $x_i^*(x_j) = \delta_{ij}$ and $\|x_i\| = 1$.

Let us describe the inductive step in detail. Suppose we have already constructed $x_1, \dots, x_k$ and $x_1^*, \dots, x_k^*$ fulfilling the above conditions. Choose $x_{k+1} \notin [x_1, \dots, x_k]$ such that $\|x_{k+1} = 1\|$. (This is possible, since X is infinite-dimensional.) The map $x_{k+1}^* \colon [x_1, \dots, x_{k+1}] \to \mathbb{R}$ given by $x_{k+1}^*(x_i) = \delta_{ij}$ is linear map on a finitely-dimensional subspace, hence it is continuous. By Hahn-Banach theorem it can be extended to a linear continuous function from X to $\mathbb{R}$.

The above conditions imply $x_k \notin \overline{[\{x_j; j \in \mathbb{N}, j \neq k\}]}$, since $x_k \notin (x_k^*)^{-1}(0)$ and the later set is a closed subspace of X containing $\{x_j; j \in \mathbb{N}, j \neq k\}$.

Now let $\mathcal{A} = \{A_i; i \in \mathbb{R}\}$ be an AD family on $\mathbb{N}$. For each $i \in \mathbb{R}$ we define

$$a_i = \sum_{j \in A_i} \frac{1}{2^j} x_j.$$

(Note that $\|\frac{1}{2^j} x_j\| \leq \frac{1}{2^j}$, which implies that the above series is Cauchy and thus convergent.)

We will show that $\{a_i; i \in \mathbb{R}\}$ is an independent set. By Theorem 1 this implies that Hamel dimension of X is at least $\mathfrak{c}$.

Let us assume that $\sum_{i \in F} c_i a_i = 0$ for some finite set F, where all c_i's are non-zero. Let

$$P := \bigcup_{\substack{i,j \in F \\ i \neq j}} (A_i \cap A_j).$$

This set is finite, since $\mathcal{A}$ is an AD family. The above finite sum can be rewritten as

$$\sum_{j=1}^{\infty} d_j x_j = 0,$$

where $d_j = \frac{c_i}{2^j}$ whenever $i \in F$ and $j \in A_i \setminus P$. Since each set $A_i \setminus P$ is infinite, we have infinitely many non-zero coefficients in this sum. Thus we can rewrite the last equation as

$$x_k = \sum_{i \neq k} f_i x_i$$

for some k and $f_i \in \mathbb{R}$, which contradicts the assumption that $x_k \notin \overline{\{x_j; j \neq k\}}$. □

[4] Or http://math.stackexchange.com/q/162387, http://math.stackexchange.com/q/278837.

Existence of unbounded linear functionals.

Proposition 2. *If X is an infinite-dimensional linear normed space, then there exist non-continuous linear function $f\colon X \to \mathbb{R}$.*

Proof. Choose an infinite independent set $\{x_n; n \in \mathbb{N}\}$ such that $\|x_n\| = 1$ for each $n \in \mathbb{N}$ and a function $f\colon X \to \mathbb{R}$ such that $f(x_n) = n$. □

Continuity of coordinate functionals. If B is a Hamel basis of a vector space X over $\mathbb{R}$, and we define $f_b\colon x \to \mathbb{R}$ which assigns to x its b-th coordinate, i.e., $x = \sum_{b\in B} f_b(x)b$ for each $x \in X$, then f_b is a linear function from X to $\mathbb{R}$.

Suppose that X is, moreover, a Banach space. We would like to know whether the functions f_b are continuous. We will show that at most finitely many of them can be continuous.

Proposition 3. *Let B be a Hamel basis of a Banach space X. Let f_b, $b \in B$, be the coordinate functionals. Then there is only finitely many b's such that f_b is continuous.*

Proof. Suppose that $\{b_i; i \in \mathbb{N}\}$ is an infinite subset of B such that each f_{b_i} is continuous. W.l.o.g. we may assume that $\|b_i\| = 1$.

Let
$$x := \sum_{i=1}^{\infty} \frac{1}{2^i} b_i.$$
(Since X is complete, the above sum converges.)

We also denote $x_n := \sum_{i=1}^{n} \frac{1}{2^i} b_i$. Since x_n converges to x, we have $f_{b_k}(x) = \lim\limits_{n\to\infty} f_{b_k}(x_n) = \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Thus the point x has infinitely many non-zero coordinates, which contradicts the definition of Hamel basis. □

We can give another proof based on Banach-Steinhaus theorem (uniform boundedness principle). We show first the following:

Lemma 2. *Let B be a Hamel basis of a Banach space X. Let f_b, $b \in B$, be the coordinate functionals. Let $C = \{b \in B; f_b$ is continuous$\}$. Then $\sup\{\|f_b\|; b \in C\} < \infty$.*

Proof. For any $x \in X$ there is at most finitely many b's in C such that $f_b(x) \neq 0$. This implies that $\sup_{b\in C}|f_b(x)|$ is finite. Banach-Steinhaus theorem this implies $\sup\{\|f_b\|; b \in C\} < \infty$. □

Proof of Proposition 3. Let B be any Hamel basis for X. For any choice of constants c_b, $b \in B$, is the set $\{c_b f_b; b \in B\}$ a Hamel basis as well. The coordinate functionals for this new basis are $g_b = \frac{1}{c_b} f_b$. If the set $C = \{b \in B; f_b$ is continuous$\}$ is infinite, then by an appropriate choice of constant c_b we can obtain $\sup\{\|f_b\|; b \in C\} = \infty$, which contradicts the above lemma. □

It is easy to show that finitely many of coordinate functionals can be continuous. If X is a Banach space with a basis B and $x_1, \dots, x_n \notin X$, then $[x_1, \dots, x_n] \oplus X$ is a Banach space with a basis $\{x_1, \dots, x_n\} \cup B$ and there are at least n continuous coordinate functionals.

Also in the space c_{00} from Example 1 with sup-norm all coordinate functionals are continuous. The space c_{00} is, of course, not complete.

Cauchy functional equation

References: [Ku, Section 5.2, Chapter 12], [S, Section 2.1], [Ka, Chapter 1], [Kh, Chapter 7], [Her, Section 5.1], [A, Appendix to Chapter 6]

Let us study the functions $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling

$$f(x + y) = f(x) + f(y). \tag{1}$$

The equation (1) is called *Cauchy equation* and functions fulfilling (1) are called *additive functions.*

It is easy to show that

Lemma 3. *If a function $f\colon \mathbb{R} \to \mathbb{R}$ fulfills (1), then*

$$f(qx) = qf(x)$$

holds for every $q \in \mathbb{Q}$, $x \in \mathbb{R}$.

This shows, that the additive functions are precisely the linear maps if we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

Lemma 3 implies that

Theorem 4. *Every continuous solution (1) is of the form $f(x) = ax$ for some $a \in \mathbb{R}$.*

Non-linear solutions

Using the existence of Hamel basis in $\mathbb{R}$ (as a vector space over $\mathbb{Q}$) we can show that

Theorem 5. *There exist non-linear solution of (1), i.e. functions $f\colon \mathbb{R} \to \mathbb{R}$ that fulfill (1) but are not of the form $f(x) = ax$.*

Theorem 6. *If f is a non-linear solution of (1), then the graph of this function*

$$G(f) = \{(x, f(x)); x \in \mathbb{R}\}$$

is dense in $\mathbb{R}^2$.

The proof can be found e.g. in [Her, Theorem 5.4].

Theorems 4 and 6 suggest that well-behaved solutions of (1) are linear and that non-linear solutions have to be, in some sense, pathological. Let us mention a one more result in this direction.

Theorem 7. *Every measurable solution of* (1) *is linear.*

An elegant proof is given in [Her, Theorem 5.5].

This last result means that by showing the existence of non-continuous solutions of (1) we have also obtained the existence of non-measurable sets.

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