Topological groups

References: [AT]

1 Some basic facts

- The map $x \mapsto x^{-1}$ is a homoemoprhism.
- For any $g \in G$ the left multiplication $x \mapsto gx$ and the right multiplication $x \mapsto xg$ are homeomorphisms.
- If $\mathcal{B}$ is base at e than $x\mathcal{B} = \{xU; U \in \mathcal{B}\}$ is base at x. (The same is true for $\mathcal{B}x$.)
- If $U \subseteq G$ is open subset and $S \subseteq G$ is arbitrary subset of G, then both US and SU are open.
- If $K, L \subseteq G$ are compact subsets of G, then KL is compact, too.
- If V is a neighborhood of e, then $V \subseteq \overline{V} \subseteq V^2$.
- For any neighborhood W of e there exists a symmetric neighborhood U such that $U^2 \subseteq W$.
- Every open subgroup is closed.
- Closure of a subgroup is a subgroup.
- Every subgroup is either clopen or has empty interior.

1.1 Separation axioms

1.2 Uniformities

Every topological group is a uniform space.

Each of these three uniformities gives the original topology.

$$
\begin{aligned}
O^l_V &= \{(g,h) \in G \times G; g^{-1}h \in V\} \\
O^r_V &= \{(g,h) \in G \times G; gh^{-1} \in V\} \\
O_V &= O^l_V \cap O^r_V
\end{aligned}
$$

[AT, Theorem 1.8.3]
TODO

2 Products of topological group

Σ-*product* is subspace of $\prod X_\alpha$ consisting of all points such that only countably many coordinates differ from the corresponding coordinates of the base point. σ-*product* = finitely many.

For topological groups: base point is identity.

σ-*product* of topological groups G_α with neutral element e_α is the subspace of $\prod G_\alpha$ consisting of those elements for which $\{\alpha; x_\alpha \neq e_\alpha\}$ is finite. (=elements with finite support)

σ-product of topological groups is a dense subgroup of $\prod G_\alpha$

[AT, Proposition 1.6.41]: The σ-product of any σ-compact spaces is σ-compact.

3 P-groups

[AT, p.249]: The classes of P-spaces and P-groups are peculiar in many respects; they may serve as a source of examples and counterexamples of topological groups with unusual combinations of properties.

[AT, Lemma 4.4.1]: If G is a P-group, then

- G has a base at identity consisting of open subgroups, so G is zero-dimensional.
- If G is ω-narrow, then it has a base of identity which consists of open invariant subgroups.
- Every (topological) quotient group of G is also a P-group.
- If G is a dense subgroup of a topological group H, then H is a P-group.

3.1 Lindelöf P-groups

[AT, Chapter 4, p.216]: Lindelöf P-space, in many respect, behave as compact Hausdorff spaces.

[AT, Lemma 4.4.2]: Let G be an ω-narrow P-group. Then every homomorphic continuous image K of G with $\psi(K) \leq \aleph_0$ is countable.

[AT, Lemma 4.4.3]: Lindelöf subspace of a Hausdorff P-space is closed.

[AT, Proposition 4.4.5]: Every Lindelöf P-group is Raikov complete.

[AT, Proposition 4.4.10]: Product of countably many Lindelöf P-spaces is Lindelöf.

4 Raikov complete groups

TODO *Cauchy filter*

TODO *Raikov complete*

[AT, Lemma 3.6.10] Every topological group has Raikov completion.

[AT, Theorem 3.6.22] Product of Raikov complete topological groups is Raikov complete.

[AT, Theorem 3.6.24] Every locally compact topological group G is Raikov complete.

[AT, Theorem 3.6.25] Raikov complete = complete w.r.t. two-sided uniformity

TODO Raikov complete ⇔ absolutely closed (H-closed)

[AT, Exercise 3.6.m]: Closed subgroup of a Raikov complete group is Raikov complete.

5 Precompact groups and precompact sets

Precompact = for every neighborhood U of identity there exists a finite subset F of G such that $G = FU$. They are also called *totally bounded.*

Precompact subset

A topological group G is compact if and only if it is precompact and Raikov complete. [AT, Theorem 3.7.15]

A topological group G is precompact if and only if its Raikov completion is compact. [AT, Theorem 3.7.16]

A topological group G is precompact if and only if it is topologically isomorphic to a subgroup of a compact group. [AT, Corollary 3.7.17]

Closure properties

- H=hereditary (subgroups)
- CH=closed subgroups
- D="closure" (If a dense subgroup has this property, so does the whole group.)

- FP=finite products
- P=arbitrary products (multiplicative)
- HI=homomorphic image

	H	CH	D	FP	P	HI
P-group						
Lindelöf P-group						

References

[AT] A. V. Arkhangelskii and M. Tkachenko. *Topological Groups and Related Structures.* Atlantis Press/World Scientific, Amsterdam, 2008.