Notes on some papers related to topological groups

1 Hernández: Topological groups close to being σ-compact

Notes from [H].

1.1 Introduction

o-bounded, *strictly o-bounded*

The game in definition of strictly o-bounded topological group is called OF-game in [HRT, Tka2, Tka1].

$\aleph_0$-*bounded* They are called ω-*narrow* in [AT].

subgroup of a σ-compact group $\Rightarrow$ strictly o-bounded $\Rightarrow$ o-bounded $\Rightarrow$ $\aleph_0$-bounded

None of the implication can be reversed

TODO $\mathbb{R}$-factorizable

1.2 Elementary properties

- Every subgroup of a (strictly) o-bounded group is (strictly) o-bounded.
- Every continuous image of (strictly) o-bounded group is (strictly) o-bounded.
- Every Lindelöf P-group is o-bounded.
- Every ω-narrow ($\aleph_0$-bounded) P-group is o-bounded. (This is strengthening of the previous result which does not appear in [H]. However, the same proof works without any change, as pointed out in [HRT].)

The author asks in Problem 2.5 whether every Lindelöf P-group is strictly o-bounded. A counterexample was given in [KM]. They constructed a Lindelöf P-group which is not strictly o-bounded.

Example 2.6: $\mathbb{R}^\omega$ is $\aleph_0$-bounded but not o-bounded.

1.3 o-bounded and σ-compact groups

Example 3.1 gives Example of a group G^* which is o-bounded but not a subgroup of a σ-compact group. The group G^* is moreover Lindelöf P-group.

This example is somewhat similar to [AT, 4.4.11]. The same example also appears in [HRT], where spaces obtained by this construction are called Comfort-like groups.

Let us describe this example in more detail.

1.3.1 Group Π (ω-box product of countable discrete groups)

We work with a family $\{G_\alpha; \alpha \in A\}$ of countable discrete groups. (In fact, to get a desired example we can take $G_\alpha = \{0,1\}$. As far as I can tell, nothing substantial changes in the proof.)

Let $\Pi = \prod\limits_{\alpha\in A} G_\alpha$ with the $\aleph_0$-box topology, i.e., the basic open sets are $p_B^{-1}(x)$, where B is a countable subset of A and $p_B\colon \Pi \to \Pi_B$ is the projection.

The local basis at e is formed by the sets $p_B^{-1}(e)$ for $B \in [A]^{\le\omega}$. (In the other words, we fix $\pi_\alpha(x) = e$ for α's from some countable set B and the remaining coordinates are arbitrary.)

Π *is a Hausdorff topological group.*

Π *is a P-group.* Obvious. (It suffices to show that a countable intersection of basic neighborhoods of e is open. We have $\bigcap\limits_{i=0}^{\infty} p_{B_i}^{-1}(e) = p_B^{-1}(e)$ for $B = \bigcup\limits_{i=0}^{\infty} B_i$.)

Π *is* $\aleph_0$*-bounded.*

Π *is o-bounded.*

1.3.2 Subgroup G^* (finite support)

For $g \in \Pi$ we define

$$\operatorname{supp} g = \{\alpha \in A; \pi_\alpha(g) \neq e_\alpha\}.$$

We put

$$G^* = \{g \in \Pi; |\operatorname{supp} g| < \aleph_0\}.$$

G^* *is a subgroup of* G. Obvious. Just notice that $\operatorname{supp}(g_1 g_2) \subseteq \operatorname{supp}(g_1) \cup \operatorname{supp}(g_2)$.

G^* *is o-bounded* since this is true for Π and both these properties are hereditary.

G^* *is Lindelöf.* TODO [C]

Basic neighborhoods. The family $\mathcal{U} = \{U_B; B \in [A]^{\leq\omega}\}$ where

$$U_B = p_B^{-1}(e) \cap G^*$$

is a local base at e.

- The sets U_B are clopen. (The complement of U_B is $\{x \in G^*; (\exists \beta \in B) \pi_\beta(x) \neq e\}$, so it is union of open sets $\pi_\beta^{-1}(G_\beta \setminus \{e\})$)
- Every U_B is a subgroup of G^*.
- $|G^*/U_B| \leq \aleph_0$ for each $B \in [A]^{\leq\omega}$. (This basically just says that G^* is $\aleph_0$-bounded.)

1.3.3 G^* is strictly o-bounded

G^* *is strictly o-bounded.*

1.3.4 G^* is not σ-compact for $|A| \geq \aleph_1$

We will show that if $|A| \geq \aleph_1$, then G^*. (I do not see this condition mentioned in [H], but I think it is needed.)

Lemma 1. *If* $K \subseteq G^*$ *is compact, then*

$$\operatorname{supp} K = \bigcup_{g \in K} \operatorname{supp} g$$

is finite.

Proof. Let us assume that $\sup K$ is infinite. Since each $g \in K$ has finite support, this implies that we can construct a sequence $g_n \in K$, $\alpha_n \in A$ such that

$$g_n(\alpha_n) \neq e,$$

where all α_n's are distinct.

If we prescribe values at each of countably many coordinates $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$, we get an open set:

$$U_n = \{g \in G^*; \pi_{\alpha_n} \neq e, \pi_{\alpha_j} = e \text{ for } j \neq n\}.$$

Moreover, $g_n \in U_n$, and $g_j \notin U_n$ for $j \neq n$.

This implies that U_n's together with the set $G^* \setminus \{g_n; n \in \omega\}$ form an open cover of G^* which has no finite subcover. (Any finite subsystem contains only finitely many g_n's.) □

Proposition 1. *If* $A \geq \aleph_1$ *then* G^* *is not* σ*-compact.*

Proof. □

1.4 A characterization of o-bounded groups

An $\aleph_0$-bounded topological group G is o-bounded if and only if all second countable continuous homomorphic images of G are o-bounded.

Problem 4.2 asks about similar characterization for strictly o-bounded groups. In [HRT, Theorem 3.1] the authors show that such characterization is not true under $\diamondsuit$.

1.5 The space $C_p(X)$ and productivity properties

The author asks in Problem 5.2 whether product of two o-bounded groups is o-bounded. The answer is no, [HRT, Example 2.12] is an example of a second countable o-bounded topological group G such that $G \times G$ is not o-bounded.

References

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