`subnets2.tex` April 22, 2009

`http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/topo/`

This text contains notes for my talk given at our topology seminar. It compares 3 different definitions of subnets. The basic material for this talk was the book [S]. I tried to use notation and terminology in accordance with this book.

1 Directed sets and nets

We first recall some basic definitions.

Definition 1.1. We say that $(\mathbb{D}, \leq)$ is a *directed set,* if $\leq$ is a relation on $\mathbb{D}$ such that

(i) $x \leq y \wedge y \leq z \Rightarrow x \leq z$ for each $x, y, z \in \mathbb{D}$;

(ii) $x \leq x$ for each $x \in \mathbb{D}$;

(iii) for each $x, y \in \mathbb{D}$ there exist $z \in \mathbb{D}$ with $x \leq z$ and $y \leq z$.

In the other words a directed set is a set with a relation which is reflexive, transitive (=preorder or quasi-order) and upwards-directed.

The following two notions will be often useful for us

Definition 1.2. A subset A of set $\mathbb{D}$ directed by $\leq$ is *cofinal* (or *frequent*) in $\mathbb{D}$ if for every $d \in \mathbb{D}$ there exists an $a \in A$ such that $d \leq a$.

A subset A of a directed set $\mathbb{D}$ is called *residual* (or *eventual*) if there is some $d_0 \in \mathbb{D}$ such that $d \geq d_0$ implies $d \in A$.

Clearly, every residual set is cofinal.

Definition 1.3. A set of the form

$$\mathbb{D}_d = \{d' \in \mathbb{D}; d' \geq d\},$$

where d is an element of a directed set $\mathbb{D}$, will be called *section* or *tail* of $\mathbb{D}$.

The set $\mathcal{B} = \{\mathbb{D}_d; d \in \mathbb{D}\}$ is clearly a filter base. The filter $\mathcal{F}$ generated by $\mathcal{B}$ is called the *section filter* of the directed set $\mathbb{D}$.

We see directly from the definition that:
A is residual $\Leftrightarrow$ A contains a section;
A is cofinal $\Leftrightarrow$ A has non-empty intersection with every section;
A is residual $\Leftrightarrow$ $\mathbb{D} \setminus A$ is not cofinal.

We will also need the notion of cofinal map.

Definition 1.4. [[R, Definition 3.3.13]] A function $f\colon P \to \mathbb{D}$ from a preordered set to a directed set is *cofinal* if for each $d_0 \in \mathbb{D}$ there exists $p_0 \in P$ such that $f(p) \geq d_0$ whenever $p \geq p_0$.

Definition 1.5. A *net* in a topological space X (or in a set X) is a map from any non-empty directed set $\mathbb{D}$ to X. It is denoted by $(x_d)_{d\in\mathbb{D}}$.

We can define the notions analogous to the notions from Definition 1.2 for nets as well.

Definition 1.6. Let $(x_d)_{d\in\mathbb{D}}$ be a net in X and let $S \subseteq X$.

If $S = \{x_d; d \geq d_0\}$ for some $d_0 \in \mathbb{D}$, then S is called *tail set* of (x_d).

S is an *eventual* (or *residual*) set of the net if S contains some tail set i.e., if there is some $d_0 \in \mathbb{D}$ such that $\{x_d; d \geq d_0\} \subseteq S$.

S is a *frequent* (or *cofinal*) set of the net if S meets every tail set i.e., if for each $d_0 \in \mathbb{D}$ there is some $d \geq d_0$ such that $x_d \in S$.

S is *infrequent* if it is not frequent.

The corresponding notions for directed sets are now special case of this definition if we consider the net $id_{\mathbb{D}} \colon \mathbb{D} \to \mathbb{D}$.

Note that:

S is eventual $\Leftrightarrow$ S contains a tail set

S is frequent $\Leftrightarrow$ S intersects each tail set

S is eventual $\Leftrightarrow$ $X \setminus S$ is infrequent

All eventual sets of (x_d) form a filter, it is called *eventuality filter* (or *section filter*) of (x_d).

Definition 1.7. A net $(x_d)_{d\in\mathbb{D}}$ in a topological space X is said to be *convergent* to $x \in X$ if for each neighborhood U of x there exists $d_0 \in \mathbb{D}$ such that $x_d \in U$ for each $d \geq d_0$.

$$(\forall_{U\in\mathcal{T}} U \ni x)(\exists d_0 \in \mathbb{D})(\forall d \geq d_0) x_d \in U$$

If a net $(x_d)_{d\in\mathbb{D}}$ converges to x, the point x is called a *limit* of this net.

The set of all limits of a net is denoted $\lim x_d$.

A net converges to x if and only if its eventuality filter converges to x.

This correspondence goes the other way round as well. To a filter $\mathcal{F}$ we assign a directed set $\{(a, A); a \in A \in \mathcal{F}\}$ ordered by

$$(a, A) \leq (b, B) \qquad \Leftrightarrow \qquad A \supseteq B$$

and a net

$$x_{a,A} = a.$$

This net converges to a point if and only if the filter $\mathcal{F}$ does and, moreover, the eventuality filter of this net is $\mathcal{F}$ again.

1.1 Prime space associated with a directed set

A viewpoint introduced in this section can be sometimes useful when dealing with nets.

Definition 1.8. To each directed set $\mathbb{D}$ we assign a topological space $P(\mathbb{D})$ on a set $\mathbb{D} \cup \{\infty\}$ (where ∞ is any point with $\infty \notin \mathbb{D}$) such that the points of $\mathbb{D}$ are isolated and the base at ∞ consists of all upper sections of $\mathbb{D}$.

This space is closely related to the convergence of a net.

Lemma 1.9. *A net $(x_d)_{d\in\mathbb{D}}$ converges to X if and only if the map $f\colon P(\mathbb{D}) \to X$ given by $f(\infty) = x$ and $f(d) = x_d$ is continuous.*

Lemma 1.10. *Let $f\colon \mathbb{D}' \to \mathbb{D}$ be a map between two directed sets. The following conditions are equivalent.*

(i) *f is a cofinal map,*

(ii) *the map $f\colon \mathbb{D}' \to P(\mathbb{D})$ is a convergent net,*

(iii) *the extension $\overline{f}\colon P(\mathbb{D}') \to P(\mathbb{D})$ is continuous,*

(iv) *$f^{-1}(M)$ is residual in $\mathbb{D}'$ whenever M is residual in $\mathbb{D}$*

(v) *if M is cofinal in $\mathbb{D}'$ then $f[M]$ is cofinal in $\mathbb{D}$*

1.2 Historical note

The notion of nets was defined by E. H. Moore and H. L. Smith and developed by many other mathematicians. It was widely popularized by Kelley's book.

According to [M, p.143]: The terminology was not Kelley's invention, though. Kelley had wanted to call such an object a way. However, nets have subnets, which Kelley would have dubbed subways. Norman Steenrod talked him out of it. After some prodding by Kelley, Steenrod suggested the term net as a substitute for way.

2 Three definitions of subnet

When trying to find a notion of subnet, which would reflect out intuition for sequences and subsequences, there are some problems. The notion directly mimicking the definition of subsequence would be the following:

Definition 2.1. If $(x_a)_{a\in\mathbb{A}}$ is a net in X and $\mathbb{B} \subseteq \mathbb{A}$ is a cofinal subset of $\mathbb{A}$, then $(x_a)_{a\in\mathbb{B}}$ is again a net in X. Every such net is called a *cofinal subnet* (or *frequent subnet*) of the net $(x_a)_{a\in\mathbb{A}}$.

Unfortunately, many results which hold for subsequences in metric spaces fail for cofinal subnets in general topological spaces (e.g., the characterization of compact spaces[1]). Therefore different definition of a subnet is needed.

We will describe three notions of a subnet, which are commonly used. We start with the most general one. It is named after Aarnes and Andenæs, who investigated it in [AA].

[1] $\beta\mathbb{N}$ is a compact space which is not sequentially compact – there are no not-trivial convergent sequences in $\beta\mathbb{N}$; [E, Corollary 3.6.15]; a different examples are given in [R, Example 3.3.22], [S, 17.28-29]

Definition 2.2. Let $(x_\alpha : \alpha \in \mathbb{A})$ and $(y_\beta : \beta \in \mathbb{B})$ be nets in a set X, with eventuality filters $\mathcal{F}$ and $\mathcal{G}$, respectively. The net $(y_\beta : \beta \in \mathbb{B})$ is an *AA subnet* of $(x_\alpha : \alpha \in \mathbb{A})$ if any of the following equivalent conditions is fulfilled:

(i) Every (y_β)-frequent subset of X is also (x_α)-frequent.

(ii) Every (x_α)-eventual subset of X is also (y_β)-eventual.

(iii) $\mathcal{G} \supseteq \mathcal{F}$

(iv) Each (x_α)-tail set contains some (y_β)-tail set.

(v) For each eventual set $\mathbb{S} \subseteq \mathbb{A}$, the set $y^{-1}(x(\mathbb{S}))$ is eventual in $\mathbb{B}$.

Note that now a cofinal map $f\colon \mathbb{D}' \to \mathbb{D}$ can be equivalently characterized as an AA-subnet of the identity map $id_{\mathbb{D}}$.

The remaining two definitions of subnet can be described using the notion of cofinal map.

Definition 2.3. Let $(x_\alpha : \alpha \in \mathbb{A})$ and $(y_\beta : \beta \in \mathbb{B})$ be nets in a set X.

If there exists a cofinal map $\varphi\colon \mathbb{B} \to \mathbb{A}$ such that $y_\beta = x_{\varphi(\beta)}$, then (y_β) is a *Kelley subnet* of (x_α). (This can be reformulated as: $y = x \circ \varphi$ for some cofinal φ.)

If there exists a map φ which is, in addition to the above conditions, monotone, then (y_β) is a *Willard subnet* of (x_α).

The following implication hold and none of them can be conversed:
frequent subnet $\Rightarrow$ Willard subnet $\Rightarrow$ Kelley subnet $\Rightarrow$ AA-subnet

We next show that the notions of Willard, Kelley and AA-subnet are in a sense "compatible" and they can be used interchangeably in most situations.

Definition 2.4. Two nets are called *AA-equivalent* if each of them is AA-subnet of another one.

Clearly, this is equivalent to saying that the two nets have the same eventuality filter.

We will need the following lemma ([S, Lemma 7.18]):

Lemma 2.5. *Let $(u_a : a \in \mathbb{A})$ and $(v_b : b \in \mathbb{B})$ be nets taking values in a set X and let $\mathcal{F}$, $\mathcal{G}$ be their eventuality filters. Then the following conditions are equivalent:*

(A) *$F \cap G$ is nonempty, for every $F \in \mathcal{F}$, $G \in \mathcal{G}$*

(B) *$\mathcal{M} = \{S \subseteq X; S \supseteq F \cap G$ for some $F \in \mathcal{F}, G \in \mathcal{G}\}$ is a proper filter.*

(C) *The filters $\mathcal{F}$ and $\mathcal{G}$ have a common proper superfilter.*

(D) *The given nets have a common AA subnet*

(E) *The given nets have a common Willard subnet, i.e., there exists a net* $(p_\lambda : \lambda \in \mathbb{L})$ *which is a Willard subnet of both given nets. Furthermore, that net can be chosen so that it is a* maximal common AA subnet *of the these nets i.e., so that if* (q_μ) *is any common AA subnet of them, then* (q_μ) *is also an AA subnet of* (p_λ)*.*

This lemma is true for any finite number of nets as well.

Corollary 2.6. *If* (y_β) *is an AA-subnet of* (x_α)*, then* (y_β) *is AA-equivalent to a Willard subnet of* (x_α)*.*

References

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