I have found at http://mathworld.wolfram.com/Maclaurin-CauchyTheorem.html the following claim:

Theorem (Maclaurin-Cauchy). *If $f(x)$ is positive and decreases to 0, then an Euler constant*

$$\gamma_f \equiv \lim_{n\to\infty} \left[\sum_{k=1}^{n} f(k) - \int_1^n f(x)dx \right]$$

can be defined.

For example, if $f(x) = 1/x$, then $\gamma_f = \gamma$ is just the usual Euler-Mascheroni constant.

I wanted to use this theorem but I didn't find any other reference except from these websites. Therefore I tried to prove it myself and I figured out that the proof is easy. (Moral of the story: Before looking for books and papers containing the theorem, I should have thought about a way to proof it.)

Proof. We estimate difference between the area under the graph of function f (corresponding to the integral) and the "steps" corresponding to the sum. Let us denote

$$a_n = \sum_{k=1}^{n} f(k) - \int_1^{n+1} f(x)dx.$$

We see that $0 \leq a_n - a_{n-1} = f(n) - \int\limits_n^{n+1} f(x)dx \leq f(n) - \int\limits_n^{n+1} f(n+1)dx = f(n) - f(n+1)$. The sequence a_n is n-th partial sum of a series of positive numbers $f(n) - \int\limits_n^{n+1} f(x)dx$. Therefore a_n is a non-negative increasing sequence. Moreover it is bounded, since $a_n \leq f(1) - f(2) + f(2) - f(3) + \ldots - f(n) = f(1) - f(n) \leq f(1)$. $\square$

Finally, I have found a proof based on the same idea (although it is written more elegant) in [**?**, p.194–195]. (Moral of the story: Websearch is as good for you as you're good in websearch.)

In some situations the above limit can be estimated using Euler summation formula in the following form

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)dt + \frac{f(0) + f(n)}{2} + \int_0^n (t - [t] - \frac{1}{2}) f'(t)dt$$

was possible as well, since the function f had continuous first derivative.

References

[1] J. Franke: *Vorlesungsskript Analysis I*, www.math.uni-bonn.de/people/franke/A1-4.ps