$\mathcal{I}$-variation – notes from various papers

Artur Bartoszewicz, Szymon Glab and Artur Wachowicz: Remarks on ideal boundedness, convergence and variation of sequences

[BGW].

Notation

The denote dual ideal to $\mathcal{F}$ by $\mathcal{F}^*$.
$M(\mathcal{I}) = x \in \ell^\infty(\mathcal{I})$, exists $K \in \mathcal{F}(\mathcal{I})$ such that $x|_K$ is monotone
$W(\mathcal{I}) =$ exists $K \in \mathcal{F}(\mathcal{I})$ such that $x|_K$ has bounded variation
$c^*(\mathcal{I}) =$ exists $K \in \mathcal{F}(\mathcal{I})$ such that $x|_K$ is convergent
$\ell^\infty(\mathcal{I}) =$ exists $K \in \mathcal{F}(\mathcal{I})$ such that $x|_K$ is bounded
$c(\mathcal{I}) =$ set of all $\mathcal{I}$-convergent sequences

$$M(\mathcal{I}) \subseteq W(\mathcal{I}) \subseteq c^*(\mathcal{I}) \subseteq c(\mathcal{I}) \subseteq \ell^\infty(\mathcal{I})$$

Inclusions

The inclusion $M(\mathcal{I}) \subseteq W(\mathcal{I})$. If $x \in M(\mathcal{I})$ then there exists $K_1 \in \mathcal{F}(\mathcal{I})$ such that $x|_{K_1}$ is bounded. There also exists $K_2 \in \mathcal{F}(\mathcal{I})$ such that $x|_{K_2}$ is monotone.

If we take $K = K_1 \cap K_2$, then $x|_K$ is both monotone and bounded and $K \in \mathcal{F}(\mathcal{I})$. Hence $x \in W(\mathcal{I})$.

Maximal ideals [FGT, Proposition 3]: $c(\mathcal{I}) = \ell^\infty(\mathcal{I}) \Leftrightarrow \mathcal{I}$ si a maximal ideal.

Proof. $\boxed{\Rightarrow}$ For any set $A \subseteq \mathbb{N}$ the characteristic sequence χ_A is $\mathcal{I}$-convergent. If it is convergent to 1, then $A \in \mathcal{I}$; if it is convergent to 0 then $\mathbb{N} \setminus A \in \mathcal{I}$. So for each subset A of $\mathbb{N}$ we have either $A \in \mathcal{I}$ or $\mathbb{N} \setminus A \in \mathcal{I}$, which means that $\mathcal{I}$ is maximal.

$\boxed{\Leftarrow}$ Let $\mathcal{I}$ be a maximal ideal and $x \in \ell^\infty(\mathcal{I})$. This means that $x|_K$ is bounded for some $K \in \mathcal{I}^*$. Let

$$y_n = \begin{cases} x_n & n \in K, \\ 0 & n \notin K. \end{cases}$$

Then y is bounded and, since $\mathcal{I}$ is maximal, it is $\mathcal{I}$-convergent.

But we have only changed the sequence x on the set $\mathbb{N} \setminus K \in \mathcal{I}^*$. This does not influence the $\mathcal{I}$-convergence. (Indeed, if

$$\{n \in \mathbb{N}; |y_n - \ell| \geq \varepsilon\} \in \mathcal{I}$$

then also

$$\{n \in \mathbb{N}; |x_n - \ell| \geq \varepsilon\} \subseteq |y_n - \ell| \geq \varepsilon\} \cup (\mathbb{N} \setminus K) \in \mathcal{I}.$$

) So we also have $x \in c(\mathcal{I})$. □

Theorem 4

Observation: We use several times that $A \notin \mathcal{F}^*$ implies that $A \cap M$ is infinite for each $M \in \mathcal{F}$.

Proof. Suppose that $A \cap M$ is finite for some $M \in \mathcal{F}$. Then

$$A = (A \cap M) \cup (A \setminus M) \subseteq^* A \setminus M \subseteq \mathbb{N} \setminus M$$

and thus $A \in \mathcal{F}^*$. $\square$

Proposition 3: Assume that a filter $\mathcal{F}$ is κ-generated for some $\kappa < \mathfrak{p}$. If $K \in [\mathbb{N}]^{\mathbb{N}}$ with $K \notin \mathcal{F}^*$, then there is $L \subset K$ such that $[L]^{\mathbb{N}} \cap \mathcal{F}^* = \emptyset$.

I would add (as is clear from the proof), that there exists an *infinite* set L with these properties.

Note that if $[L]^{\mathbb{N}} \cap \mathcal{F}^* = \emptyset$ for an infinite set L, then L has infinite intersection with every set from F. (Suppose that for some $F \in \mathcal{F}$ the intersection $F \cap L$ is finite. Then $L \setminus F \in \mathcal{F}^*$ (since it is a subset of $\mathbb{N} \setminus F \in \mathcal{F}^*$) and it is an infinite set.) This means that there exists a filter containing $\mathcal{F} \cup \{L\}$.

So for the set L from Proposition 3 we know that the filter $\langle \mathcal{F}, L \rangle$ generated by $\mathcal{F} \cup \{L\}$ can be created.

Proof of Proposition 3. In the proof of Proposition 3 we need that for an infinite set A we have

$$(\forall \alpha < \kappa) A \subseteq^* K \cap A_\alpha \qquad \Rightarrow \qquad A \notin \mathcal{F}^*.$$

If $A \in \mathcal{F}^*$, that means that $\mathbb{N} \setminus A \in \mathcal{F}$, i.e.,

$$\bigcap_{\alpha \in F} A_\alpha \subseteq^* \mathbb{N} \setminus A$$

for some finite set F. Using this we get

$$A \subseteq^* K \cap \bigcap_{\alpha \in F} A_\alpha \subseteq^* \mathbb{N} \setminus A$$

and

$$A \subseteq^* \mathbb{N} \setminus A.$$

This implies that $A = A \setminus (\mathbb{N} \setminus A)$ is finite, a contradiction.

Theorem 4: Assume that $\mathfrak{p} = \mathfrak{c}$. Let $\tau < \mathfrak{p}$. Suppose that $\mathcal{B}_1$, $\mathcal{B}_2$ are two properties of sequences $\mathbb{R}^{\mathbb{N}}$ such that:

(a) for all $x \in \mathbb{R}^{\mathbb{N}}$ and $K \in [\mathbb{N}]^{\mathbb{N}}$, if $x|_K$ has $\mathcal{B}_1$, then there is $L \in [\mathbb{N}]^{\mathbb{N}}$, $L \subset K$, such that $x|_L$ has $\mathcal{B}_2$;

(b) $\mathcal{B}_1$ is closed under taking subsequences, i.e., for all $x \in \mathbb{R}^{\mathbb{N}}$, $L, K \in [\mathbb{N}]^{\mathbb{N}}$, if $L \subset K$ and $x|_K$ has $\mathcal{B}_1$, then $x|_L$ has $\mathcal{B}_1$.

If a filter $\mathcal{F}$ is τ-generated, then $\mathcal{F}$ can be extended to a filter $\mathcal{F}'$ such that for any $x \in \mathbb{R}^{\mathbb{N}}$ and $K \in \mathcal{F}'$, if $x|_K$ has $\mathcal{B}_1$, then there is $L \in \mathcal{F}'$, $L \subset K$, such that $x|_L$ has $\mathcal{B}_2$.

More detailed proof of Theorem 4. We have an enumeration $\{(x_\alpha, K_\alpha); \alpha < \mathfrak{c}\}$ of all pairs such that $x|_K$ has $\mathcal{B}_1$. We want to get somehow a filter $\mathcal{F}'$ such that

$$(\forall\alpha < \mathfrak{c}) K_\alpha \in \mathcal{F}' \Rightarrow [(\exists L \subset K_\alpha)(L \in \mathcal{F}' \text{ and } x_\alpha|_L \text{ has } \mathcal{B}_2)].$$

In the other words,

$$(\forall\alpha < \mathfrak{c}) K_\alpha \notin \mathcal{F}' \vee [(\exists L \subset K_\alpha)(L \in \mathcal{F}' \text{ and } x_\alpha|_L \text{ has } \mathcal{B}_2)]. \tag{1}$$ {EQWANT}

By transfinite induction an increasing sequence $(\mathcal{F}_\alpha)_{\alpha<\mathfrak{c}}$ of filters is constructed. Then $\mathcal{F}' = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$.

In the α-th step:

If $K_\alpha \in \left(\bigcup_{\gamma<\alpha} \mathcal{F}_\alpha\right)^*$, then a set L is chosen in such way that $L \subseteq \mathbb{N} \setminus K_\alpha$. Since $L \in \mathcal{F}'$, we get that $\mathbb{N} \setminus K_\alpha \in \mathcal{F}'$ and $K_\alpha \notin \mathcal{F}'$. So in this case (1) is fulfilled.

If $K_\alpha \notin \left(\bigcup_{\gamma<\alpha} \mathcal{F}_\alpha\right)^*$ then $L' \subseteq K_\alpha$ is chosen is such way that $x_\alpha|_{L'}$ has $\mathcal{B}_2$. So in this case (1) is fulfilled too.

The ideal $\mathcal{I}_d$

The authors claim that from [LV, Theorem 1] it follows that $\mathcal{I}_d^*$ is not τ-generated for any $\tau < \mathfrak{d}$. How does it follow from that result?

References

[BGW] Artur Bartoszewicz, Szymon Glab, and Artur Wachowicz. Remarks on ideal boundedness, convergence and variation of sequences. *J. Math. Anal. Appl.*, 375(2):431–435, 2011.

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