Boos: Classical and modern methods in summability

Notes from [B].

Part I: Classical methods in summability and applications

1 Convergence and divergence

1.1 The early history of summability–the devil's invention

1.2 Summability methods: definition and examples

Definition (matrix method). (1.2.12) $\omega_A := \{x \in \omega | Ax \text{ exists}\}$ (=all series[1] $\sum_k a_{nk}x_k$ converge) = *application domain* of A
$c_A = A^{-1}(c)$ =*domain of* A

$c_0A = A^{-1}(c_0)$ = *null domain*
$m \cap c_A$ = *bounded domain*

Example. (1.2.13) (d) $B_1^* := \left(e^{-n}\frac{n^k}{k!}\right)$ *Borel matrix, discrete Borel method*

almost convergence introduced in 1948 by G. G. Lorentz [L]

$$\frac{1}{n+1}\sum_{k=p}^{p+n} x_k \overset{n\to\infty}{\longrightarrow} a \text{ (uniformly for } p)$$

f =almost convergent sequences

Theorem (almost convergence). *(1.2.18)*
(a) $c \subsetneq f \subsetneq m$ *and* $F-\lim|_c = \lim$
(b) If A is a matrix method, then $f \neq c_A$ (that is, almost convergence is not representable by a matrix method) and, moreover, $f \neq m \cap c_A$

Proof: See [L, Theorem 11]

1.3 Questions and basic notions

Inclusion theorems *conservative for null sequences* if $c_0 \subset N_V$,
conservative if $c \subset N_V$,
strongly conservative if $f \subset N_V$

[1] x is a column vector

$\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$
$\mathbb{N}^0 = \mathbb{N} \cup \{0\}$
$\omega := \mathbb{K}^{\mathbb{N}^0}$ denotes the *set of all sequences*
$m = \ell^\infty =$ bounded sequences
$c =$ convergent sequence
$c_0 = \{x = (x_k) \in c | \lim\limits_{k\to\infty} x_k = 0\} =$ the set of all *null sequences*
bs=*sequences with bounded partial sums*; $\|x\|_{bs} = \sup_n |\sum_{k=0}^n x_k|$ is called *bs-norm*
cs =*summable sequences* ($\sum_k x_k$ converges)
$\ell = \ell^1 =$ *absolutely summable sequences*
$bv =$ sequences with bounded variation
$\varphi =$ *finitely non-zero sequences*

$$\varphi \subsetneq \ell \subsetneq cs \subsetneq c_0 \subsetneq c = c_0 \oplus \langle e\rangle \subsetneq m \subsetneq \omega$$
$$\ell \subsetneq bv_0 \subsetneq bv = bv_0 \oplus \langle e\rangle \subsetneq$$

where $e := (1, 1, 1, \dots)$
$\chi = \{0, 1\}^{\mathbb{N}^0}$
$m_0 = \langle\chi\rangle = \{x = (x_k) \in \omega | \{x_k | k \in \mathbb{N}^0\}$ is a finite set$\}$
A sequence $x = (x_k) \in \chi$ is called *thin* if there exists an index sequences (k_ν) with $k_{\nu+1} - k_\nu \to \infty$
$\tau =$ *thin sequences*[2]

$$\varphi \cap \chi \subsetneq \tau \subsetneq \chi \subsetneq \langle\chi\rangle = m_0$$

Exercise 1.3.11: Borel method is regular. Borel matrix B_1^* is also regular.

2 Matrix methods: basic classical theory

2.1 Dealing with infinite series

Theorem (Abel's partial summation formula). *(2.1.1) Let $(a_\nu), (b_\nu) \in \omega$, $x_n := \sum_{\nu=0}^n a_\nu$ $(n \in \mathbb{N}^0)$ and $x_{-1} := 0$. Then the equility*

$$\sum_{\nu=n}^{n+k} a_\nu b_\nu = \sum_{\nu=n}^{n+k} x_\nu (b_\nu - b_{\nu+1}) - x_{n-1} b_n + x_{n+k} b_{n+k+1} \tag{1}$$

holds for all $n, k \in \mathbb{N}^0$. If $(x_\nu b_{\nu+1}) \in c$, the series $\sum_\nu a_\nu b_\nu$ converges if and only if the series $\sum_\nu x_\nu (b_\nu - b_{\nu+1})$ does, that is

$$(a_\nu b_\nu) \in cs \qquad \Leftrightarrow \qquad (x_\nu (b_\nu - b_{\nu+1})) \in cs$$

[2] Book says $\varphi \subseteq \tau$ – a mistake.

Corollary. *(2.1.8) A series converges absolutely if and only if each rearrangement of it is convergent.* [3]

Bibliography: [K]

2.2 Dealing with infinite matrices

Definition (products). (2.2.2)

$$yx = \sum_k y_k x_k$$

scalar product of sequences

$$Ax = \left(\sum_k a_{nk} x_k\right)_{n\in\mathbb{N}^0} \qquad yB = \left(\sum_n y_n b_{nk}\right)_{k\in\mathbb{N}^0}$$

product of a matrix and a sequence

$$AB := (c_{nk}) \text{ where } c_{nk} = \sum_\nu a_{n\nu} b_{\nu k}$$

Theorem (associativity of $t(Bx)$). *(2.2.4) Let B be an infinite matrix and $x = (x_k)$, $t = (t_k) \in \omega$. If*

(i) $x \in \omega_B$ *and* $t \in \varphi$ *or*

(ii) $t \in \ell$ *and* $\|B\| := \sup_\mu \sum_\nu |b_{\mu\nu}| < \infty$

is valid, then $(tB)x$ exists and $t(Bx) = (tB)x$ holds.

Theorem (associativity of $A(Bx)$). *(2.2.5) Let A and B be infinite matrices and $x = (x_k) \in \omega$. If*

(i) $x \in \omega_B$ *and* $A = (a_{nk})$ *is* row-finite (*that is,* $(a_{nk})_k \in \varphi$ *for each* $n \in \mathbb{N}^0$) *or*

(ii) $x \in m$, $\|B\| < \infty$ *and* $(a_{nk} \in \ell)$ *for each* $n \in \mathbb{N}^0$

holds, then $A(Bx)$ and $(AB)x$ exist and $A(Bx) = (AB)x$.

Theorem (associativity of $A(BC)$). *(2.2.6) Let A, B and C be infinite matrices. If*

(i) *BC is defined and A is row-finite or*

(ii) $\|B\| < \infty$, $(c_{\nu k})_\nu \in m$ $(k \in \mathbb{N}^0)$ *and* $(a_{n\nu})_\nu \in \ell$ $(n \in \mathbb{N}^0)$

holds, then $A(BC)$ and $(AB)C$ exist and $A(BC) = (AB)C$

[3] My question: For real series the absolute convergence is equivalent to convergence of $\sum \varepsilon_n x_n$ for any choice of $\varepsilon_n \in \{\pm 1\}$. Is there a similar theorem for complex series?

Definition (inverse). (2.2.7) Let A and B be infinite matrices. If AB exists and $AB = I$, then A is called a *left inverse of* B and B is called a *right inverse of* A. If in addition BA exists and $AB = BA = I$ holds, then the matrix B is called *bi-inverse* or simply *inverse of* A. The inverse of A, if it exists, is denoted by A^{-1}.

triangle = (lower) triangular matrix

Theorem (triangle). *(2.2.9) If A is a triangle, then the following statement hold:*

(a) *For each $y \in \omega$ there exists a unique solution of the system of equations $Ax = y$.*

(b) *There exists a unique right inverse B of A. Moreover, B is also a triangle and a left inverse. So A^{-1} exists.*

(c) *The matrix A may have more than one left inverse, but there is exactly one that is also a triangle, namely A^{-1}.*

$$\Sigma := \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \Sigma^{-1} = \begin{pmatrix} 1 & & & 0 \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

summation matrix
Bibliography: [C, ZB]

2.3 Conservative matrix methods

Definition (convergence factor sequence, β-dual). (2.3.1) For $X \subset \omega$ with $X \neq 0$ we define

$$X^\beta := \{t = (t_k) \in \omega | \forall x = (x_k) \in X \colon tx := (t_k x_k) \in cs\}.$$

Then X^β is called the *β-dual of* X or the *set of all convergence factor sequences of* X (in cs). For $y \in \omega$ we write y^β instead of $\{y\}^\beta$.

X^β is a sequence space with $\varphi < X^\beta < \omega$
$X \subset Y \subset \omega \Rightarrow Y^\beta < X^\beta$
$X \subset X^{\beta\beta} := (X^\beta)^\beta$
$\varphi^\beta = \omega$ and $\omega^\beta = \varphi$
$X \subset \omega_A \Leftrightarrow \forall n \in \mathbb{N}^0 \ \forall x = (x_k) \in X : \sum_k a_{nk} x_k$ converges $\Leftrightarrow \forall n \in \mathbb{N}^0 : (a_{nk})_k \in X^\beta$

Theorem. *(2.3.3)*

(a) $c_0^\beta = c^\beta = m^\beta = \ell$

(b) $\ell^\beta = m$

(c) $\tau^\beta = \chi^\beta = m_0^\beta = \ell$

Corollary. *(2.3.4) Let $A = (a_{nk})$ be any infinite matrix. Then*

$$\sum_k |a_{nk}| < \infty, \text{ that is } (a_{nk})_k \in \ell \text{ for each } n \in \mathbb{N}^0$$

if and only if one (thus each) of the inclusions $m \subset \omega_A$, $c \subset \omega_A$, $c_0 \subset \omega_A$, $m_0 \subset \omega_A$, and $\tau \subset \omega_A$ holds.

$$m_A := \{x \in \omega_A | Ax \in m\}$$

Theorem ($c \subset m_A$). *(2.3.5) For any matrix $A = (a_{nk})$ the following statements are equvivalent*

(a) $m \subset m_A$*, that is* $m \subset \omega_A$ *and* $A(m) \subset m$,

(b) $c \subset m_A$*, that is* $c \subset \omega_A$ *and* $A(c) \subset m$,

(c) $c_0 \subset m_A$*, that is* $c \subset \omega_A$ *and* $A(c_0) \subset m$,

(d) $\|A\| := \sup_n \sum_k |a_{nk}| < +\infty$.

Although this result can be shown easily using uniform boundedness principle, the proof used in this book illustrates "gliding hump" argument. This type of arguments is often used in functional analysis and also in connection with matrix methods [S]. (In Theorem 7.4.7 the proof of the same fact is given, using UBP.)

Theorem (conservative, regular for null sequences). *(2.3.6) Let $A = (a_{nk})$ be an infinite matrix.*

I. The following statements are equivalent:

(a) *A is conservative for null sequences (that is,* $c_0 \subset c_A$*).*

(b) $c_0 \subset \omega_A$ *and* $A(c_0) \subset c$.

(c) *A satisfies*
(Zn) $\|A\| < \infty$ *(row norm condition) and*
(Sp) $\forall k \in \mathbb{N}^0 \colon a_k = \lim_n a_{nk}$ *exists (column condition).*

LIMIT FORMULA: If A is conservative for null sequences then

$$(a_k) \in \ell \qquad \text{and} \qquad \lim_A x = \sum_k a_k x_k \qquad (x = (x_k) \in c_0).$$

II. The following statements are equivalent:

(a) *A is regular for null sequences (that is,* $c_0 \subset c_A$ *and* $\lim_A |_{c_0} = 0$*).*

(b) $c_0 \subset \omega_A$ *and* $A(c_0) \subset c_0$.

(c) *A satisfies* (Zn) *with* $a_k = 0$ *for all* $k \in \mathbb{N}^0$, *that is* (Sp$_0$) $\forall k \in \mathbb{N}^0 \colon (a_{nk})_k \in c_0$.

Theorem (of Toeplitz, Silverman, Kojima and Schur). *(2.3.7) Let* $A = (a_{nk})$ *be an infinite matrix.*

I. *The following statements are equivalent:*

(a) *A is conservative* (*that is,* $c \subset c_A$).

(b) $c \subset \omega_A$ *and* $A(c) \subset c$.

(c) *A satisfies* (Zn), (Sp) *and*
(Zs) $\sum_k a_{nk}$ $(n \in \mathbb{N}^0)$ *and* $a := \lim_n \sum_k a_{nk}$ *exist* (*row sum condition*).

LIMIT FORMULA: *If A is conservative, then*

$$\lim_A x = \chi(A) \lim x + \sum_k a_k x_k \qquad (x = (x_k) \in c),$$

where

$$\chi(A) := \lim_A e - \sum_k \lim_A e^k = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} = a - \sum_k a_k$$

is called the characteristic of A.

II. *The following statements are equivalent:*

(a) *A is regular* (*that is,* $c \subset c_A$ *and* $\lim_A|_c = \lim$).

(b) *A is regular for null sequences and* $e \in c_A$ *with* $\lim_A e = 1$.

(c) *A satisfies* (Zn), (Sp$_0$) *and the condition* (Zs$_1$), *that is* (Zs) *with* $a = 1$.

Moreover, if A is regular, then $\chi(A) = 1$.

Theorem. *(2.3.8) If A is a matrix which sums all thin sequences, then A is conservative for null sequences, that is* $\mathcal{T} \subset c_A$ *implies* $c_0 \subset c_A$ *and, in particular,* $\|A\| < \infty$. *Moreover, A is conservative, if* $\mathcal{T} \cup \{e\} \subset c_A$.

Definition. (2.3.9) A conservative matrix A, and the corresponding matrix method, is called *coregular* if $\chi(A) \neq 0$ and *conull* if $\chi(A) = 0$.

Bibliography: [H, K, P, W]

2.4 Coercive and strongly conservative matrix methods

Theorem (Schur). *(2.4.1) For a matrix* $A = (a_{nk})$ *the following statements are equivalent:*

(a) *A is coercive, that is* $m \subset c_A$.

(b) *A satisfies* (Sp), *and* $\sum_k |a_{nk}|$ *converge uniformly for* $n \in \mathbb{N}^0$.

(c) $c_0 \subset c_A$ *and*[4] $h(A) := \limsup_n \sum_k |a_{nk} - a_k| = 0$.

Corollary. *(2.4.2) Every coercive matrix is conull. In particular, a matrix cannot be both regular and coercive.*

Theorem (Hahn). *(2.4.5) If a matrix A sums all sequences of zeros and ones, then it sums all bounded sequences. That is, $\chi \subset c_A$ implies $m \subset c_A$.*

Definition (strong regularity). (2.4.6) A summability method $\mathcal{V} = (V, N_V, V-\lim)$ is called *strongly regular* if $\mathcal{V}$ is strongly conservative (that is, $f \subset N_V$) and $V - \lim|_f = F - \lim$. A matrix is called *strongly regular* if the corresponding matrix method is strongly regular.

Theorem (Lorentz). *(2.4.9) Let $A = (a_{nk})$ be a conservative matrix and a_k be the limit of the k^{th} column of A. Then the following statements are equivalent:*

(a) *A is strongly conservative.*

(b) $\limsup_n \sum_k |a_{nk} - a_{n,k+1} - a_k + a_{k+1}| = 0$.

LIMIT FORMULA: *If A is strongly conservative, then*

$$\lim_A x = \chi(A) \cdot F - \lim x + \sum_k a_k x_k \qquad (x = (x_k) \in f).$$

$f_0 := \{x \in f; F - \lim x = 0\}$

Lemma. *$bs \subset f_0$*

Theorem. *(2.4.12) If A is any matrix then the following statements are equivalent:*

(a) *A is strongly regular.*

(b) *A is regular and strongly conservative.*

(c) *A is regular and satisfies* $\limsup_n \sum_k |a_{n,k} - a_{n,k+1}| = 0$.

Example. (2.4.13) The discrete Borel method B_1^* is strongly regular.[5]

Bibliography: [L, P, R, W, ZB]

[4] Note that $h(A)$ is defined for each matrix A being conservative for null sequences.

[5] The proof of [B] claims $e^{-n} \sum_{k=n}^{\infty} \frac{n^k}{k!}$ converges to 0. This is not correct. TODO Try to find a correct proof

2.5 Abundance within domains; factor sequences

Definition (factor sequence). (2.5.1) Let A be a matrix, let $x = (x_k) \in c_A$ and $y = (y_k) \in \omega$. Then y is called a *factor sequences for x and A*, if $yx = (y_k x_k) \in c_A$.

Theorem. *(2.5.3) Let $A = (a_{nk})$ be a matrix which satisfies* (Sp). *Let a_k denote the limit of the k^{th} column of A and let $x = (x_k) \in c_A$ be given such that $\sum_k a_k x_k$ converges. If*

$$\lim_A x = \sum_k a_k x_k \tag{2}$$

and

$$\sup_{\nu,n} \left| \sum_{k=0}^{\nu} a_{nk} x_k \right| < \infty \tag{3}$$

the there exists an index sequence (r_j) such that each sequence $y = (y_k)$ which satisfies

$$\sum_{k=r_j+1}^{r_{j+1}} |y_k - y_{k-1}| \longrightarrow 0 \qquad (j \to \infty) \tag{$\mathfrak{O}$}$$

is a factor sequence for x and A. Moreover,

$$\sum_k a_k y_k x_k \text{ converges and } \lim_A yx = \sum_k a_k y_k x_k.$$

A sequence which satisfies the condition ($\mathfrak{O}\mathfrak{D}$) for some index sequence $r = (r_j)$ is called slowly oscillating (with respect to r).

Definition. (2.5.4) More generally than in 2.3.7I we define

$$\chi(G) := \lim_n \sum_k g_{nk} - \sum_k \lim_n g_{nk} = g - \sum g_k$$

for any matrix $G = (g_{nk})$ that satisfies the conditions (ZS) and (Sp) and for which $\sum_k g_k$ converges.

2.6 Comparison and consistency theorems

2.7 Triangles of type M

3 Special summability methods

4 Tauberian theorems

5 Application of boundary methods

Part II: Functional analytic methods in summability

[6]

6 Functional analytic basis

6.1 Topological spaces

6.2 Semi-metric spaces

6.3 Semi-normed spaces, Banach spaces

6.4 Locally convex spaces

Example. (6.4.15) TODO (definition of τ_ω)

6.5 Continuous linear maps and the dual space of a locally convex spaces

6.6 Dual pairs and compatible topologies

6.7 Fréchet spaces

A complete metrizable locally convex spaces is called a *Fréchet space,* or an *F-space.*

[6] My note: Also the book [M] deals with summability, e.g. in Section 2.4 *Application of Basic Principles.*

7 Topological sequences spaces: K- and FK-spaces

7.1 Sequence spaces and their ξ-duals

7.2 K-spaces

K-spaces = locally convex topologies that are stronger than τ_ω. In such stronger topologies, convergence implies coordinatewise convergence.

Definition (K-space). (7.2.2) A locally convex space (X,τ) is called a *K-space* if $X < \omega$ and $\tau_\omega \subset \tau$. In such a case τ is called a K-topology on X.

7.3 FK-spaces

Definition (FK- and BK-space). A locally convex space (X,τ) is called an *FK-space* and τ is called an *FK-topology* if (X,τ) is both, a K-space and an F-space. By definition, a *BK-space* is a normable FK-space and its topology is called BK-topology.

Example (7.3.2). (a) (ω,τ_ω) is an FK-space but no BK-space
(b) m, c, c_0, f_0 and f endowed with $\|\cdot\|_\infty$ are BK-spaces.

Definition (matrix map). (7.3.6) Let X and Y be a sequence spaces over $\mathbb{K}$, and let $T\colon X \to Y$ be a linear map. Then T is called a *matrix map* if there exists a matrix $A = (a_{nk})$ such that $X \subset \omega_A$ and $T(x) = Ax$ for all $x \in X$. (Where no confusion can arise, we denote both the matrix map and the matrix by the same letter.)

Corollary. *(7.3.7) Matrix maps between FK-spaces are continuous.*[7]

Theorem (subspace). *(7.3.8) Every closed subspace of an FK-space (endowed with the subspace topology) is an FK-space.*

7.4 Functional analytic proofs of some Toeplitz-Silverman-type theorems

Theorem ($c \subset m_A$). *(7.4.2=2.3.5) For any matrix $A = (a_{nk})$ the following statements are equvivalent*

(a) *$m \subset m_A$, that is $m \subset \omega_A$ and $A(m) \subset m$,*

(b) *$c \subset m_A$, that is $c \subset \omega_A$ and $A(c) \subset m$,*

(c) *$c_0 \subset m_A$, that is $c \subset \omega_A$ and $A(c_0) \subset m$,*

[7] My note: Summation matrix Σ defines a matrix map $\ell \to m$ or $c_\Sigma = cs \to m$. If I take sup-norm on the domain, it is not continuous. However, the subspaces ℓ and cs are not closed in the sup-norm; just note that if $x^{(k)}$ is defined by $x_n^{(k)} = \begin{cases} \frac{1}{n}; & n \le k \\ \frac{1}{n^2}; & n > k \end{cases}$ then $x^{(k)} \longrightarrow (\frac{1}{n})$ in m.

(d) $\|A\| := \sup_n \sum_k |a_{nk}| < +\infty$.

The authors provide two proofs. The first one is based on the uniform boundedness principle and the second one on Theorem 7.3.7. They also show $\|A\|$[8] $= \|A\|_{X,m}$ for $X \in \{c_0, c, m\}$.

8 Matrix methods: structure of the domains

Part III: Combining classical and functional analytic methods

9 Consistency of matrix methods

10 Saks spaces and bounded domains

11 Some aspect of topological sequence spaces

11.1 An inclusion theorem

11.2 Gliding hump and oscillating properties

11.3 Theorems of Toeplitz-Silverman type via sectional convergence and

11.4 Barelled K-spaces

11.5 The sequences of zeros and ones in a sequences spaces

Contents

[8] TODO Check in the book where they define $\|A\|_{X,Y}$. I think it is the norm of Theorem 6.3.19, althought they do not use this notation in that theorem.

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