Boos: Classical and modern methods in summability

Notes from [B].

Part I: Classical methods in summability and applications

1 Convergence and divergence

1.1 The early history of summability—the devil’s invention

1.2 Summability methods: definition and examples

Definition (matrix method). (1.2.12) $\omega_A := \{ x \in \omega | Ax \text{ exists} \}$ (= all series of $\sum_k a_{nk} x_k$ converge) = application domain of $A$

$c_A = A^{-1}(c)$ = domain of $A$

$c_0A = A^{-1}(c_0)$ = null domain

$m \cap c_A = $ bounded domain

Example. (1.2.13) (d) $B^*_1 := \left( e^{-n \frac{n^2}{n^2}} \right)$ Borel matrix, discrete Borel method

almost convergence introduced in 1948 by G. G. Lorentz [L]

\[
\frac{1}{n+1} \sum_{k=p}^{n} x_k \xrightarrow{n \to \infty} a \quad \text{(uniformly for } p)\]

$f =$ almost convergent sequences

Theorem (almost convergence). (1.2.18)

(a) $c \subseteq f \subseteq m$ and $F - \lim |c| = \lim$

(b) If $A$ is a matrix method, then $f \neq c_A$ (that is, almost convergence is not representable by a matrix method) and, moreover, $f \neq m \cap c_A$

Proof: See [L, Theorem 11]

1.3 Questions and basic notions

Inclusion theorems

conservative for null sequences if $c_0 \subseteq N_V$,

conservative if $c \subseteq N_V$,

strongly conservative if $f \subseteq N_V$

$^1x$ is a column vector
\[ K = \mathbb{C} \text{ or } K = \mathbb{R} \]
\[ N^0 = \mathbb{N} \cup \{0\} \]
\[ \omega := \mathbb{R}^{N^0} \text{ denotes the set of all sequences} \]
\[ m = \ell^\infty = \text{bounded sequences} \]
\[ c = \text{convergent sequence} \]
\[ c_0 = \{ x = (x_k) \in c \mid \lim_{k \to \infty} x_k = 0 \} = \text{the set of all null sequences} \]
\[ bs = \text{sequences with bounded partial sums}; \|x\|_{bs} = \sup_n |\sum_{k=0}^n x_k| \text{ is called } bs-norm \]
\[ cs = \text{summable sequences (} \sum x_k \text{ converges)} \]
\[ \ell = \ell^1 = \text{absolutely summable sequences} \]
\[ bv = \text{sequences with bounded variation} \]
\[ \varphi = \text{finitely non-zero sequences} \]

\[ \varphi \subseteq \ell \subseteq cs \subseteq c_0 \subseteq c = c_0 \oplus \langle e \rangle \subseteq m \subseteq \omega \]
\[ \ell \subseteq bv_0 \subseteq bv = bv_0 \oplus \langle e \rangle \subseteq \omega \]

where \( e := (1, 1, 1, \ldots) \)
\[ \chi = \{0, 1\}^{N^0} \]
\[ m_0 = \langle \chi \rangle = \{ x = (x_k) \in \omega \mid \{x_k\}_{k \in N^0} \text{ is a finite set} \} \]
A sequence \( x = (x_k) \in \chi \) is called thin if there exists an index sequences \( (k_\nu) \) with \( k_{\nu+1} - k_\nu \to \infty \)
\[ \tau = \text{thin sequences}^2 \]

\[ \varphi \cap \chi \subseteq \tau \subseteq \chi \subseteq \langle \chi \rangle = m_0 \]

Exercise 1.3.11: Borel method is regular. Borel matrix \( B_1^* \) is also regular.

## 2 Matrix methods: basic classical theory

### 2.1 Dealing with infinite series

**Theorem (Abel’s partial summation formula).** (2.1.1) Let \((a_\nu), (b_\nu) \in \omega, x_n := \sum_{\nu=0}^n a_\nu\) \((n \in N^0) \) and \(x_{-1} := 0\). Then the equality

\[
\sum_{\nu=n}^{n+k} a_\nu b_\nu = \sum_{\nu=n}^{n+k} x_\nu (b_\nu - b_{\nu+1}) - x_{n-1} b_n + x_{n+k} b_{n+k+1} \tag{1}
\]

holds for all \(n, k \in N^0\). If \((x_\nu b_{\nu+1}) \in c\), the series \( \sum_\nu x_\nu (b_\nu - b_{\nu+1}) \) converges if and only if the series \( \sum_\nu (a_\nu b_\nu) \) converges, that is

\[
(a_\nu b_\nu) \in cs \iff (x_\nu (b_\nu - b_{\nu+1})) \in cs
\]

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^2 Book says \( \varphi \subseteq \tau \) – a mistake.
Corollary. (2.1.8) A series converges absolutely if and only if each rearrangement of it is convergent.  

Bibliography: [K]

2.2 Dealing with infinite matrices

Definition (products). (2.2.2)

\[ yx = \sum_k y_k x_k \]

scalar product of sequences

\[ Ax = \left( \sum_k a_{nk} x_k \right)_{n \in \mathbb{N}^0} \quad yB = \left( \sum_n y_n b_{nk} \right)_{k \in \mathbb{N}^0} \]

product of a matrix and a sequence

\[ AB := (c_{nk}) \text{ where } c_{nk} = \sum_\nu a_{n\nu} b_{\nu k} \]

Theorem (associativity of \(t(Bx)\)). (2.2.4) Let \(B\) be an infinite matrix and \(x = (x_k), t = (t_k) \in \omega\). If

(i) \(x \in \omega_B\) and \(t \in \nu\) or

(ii) \(t \in \ell\) and \(\|B\| := \sup_\mu \sum_\nu |b_{\mu \nu}| < \infty\)

is valid, then \((tB)x\) exists and \(t(Bx) = (tB)x\) holds.

Theorem (associativity of \(A(Bx)\)). (2.2.5) Let \(A\) and \(B\) be infinite matrices and \(x = (x_k) \in \omega\). If

(i) \(x \in \omega_B\) and \(A = (a_{nk})\) is row-finite (that is, \((a_{nk})_k \in \nu\) for each \(n \in \mathbb{N}^0\)) or

(ii) \(x \in m\), \(\|B\| < \infty\) and \((a_{nk})_k \in \ell\) for each \(n \in \mathbb{N}^0\)

holds, then \(A(Bx)\) and \((AB)x\) exist and \(A(Bx) = (AB)x\).

Theorem (associativity of \(A(BC)\)). (2.2.6) Let \(A\), \(B\) and \(C\) be infinite matrices. If

(i) \(BC\) is defined and \(A\) is row-finite or

(ii) \(\|B\| < \infty\), \((c_{nk})_\nu \in m\) \((k \in \mathbb{N}^0)\) and \((a_{nu})_n \in \ell\) \((n \in \mathbb{N}^0)\)

holds, then \(A(BC)\) and \((AB)C\) exist and \(A(BC) = (AB)C\)

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3My question: For real series the absolute convergence is equivalent to convergence of \(\sum \varepsilon_n x_n\) for any choice of \(\varepsilon_n \in \{\pm 1\}\). Is there a similar theorem for complex series?
**Definition (inverse).** (2.2.7) Let $A$ and $B$ be infinite matrices. If $AB$ exists and $AB = I$, then $A$ is called a left inverse of $B$ and $B$ is called a right inverse of $A$. If in addition $BA$ exists and $AB = BA = I$ holds, then the matrix $B$ is called bi-inverse or simply inverse of $A$. The inverse of $A$, if it exists, is denoted by $A^{-1}$.

triangle = (lower) triangular matrix

**Theorem (triangle).** (2.2.9) If $A$ is a triangle, then the following statement hold:

(a) For each $y \in \omega$ there exists a unique solution of the system of equations $Ax = y$.

(b) There exists a unique right inverse $B$ of $A$. Moreover, $B$ is also a triangle and a left inverse. So $A^{-1}$ exists.

(c) The matrix $A$ may have more than one left inverse, but there is exactly one that is also a triangle, namely $A^{-1}$.

\[
\Sigma := \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ \vdots & \ddots & \ddots \end{pmatrix}
\]

summation matrix

Bibliography: [C, ZB]

2.3 Conservative matrix methods

**Definition (convergence factor sequence, $\beta$-dual).** (2.3.1) For $X \subset \omega$ with $X \neq 0$ we define

\[
X^\beta := \{ t = (t_k) \in \omega | \forall x = (x_k) \in X : tx = (t_kx_k) \in cs \}.
\]

Then $X^\beta$ is called the $\beta$-dual of $X$ or the set of all convergence factor sequences of $X$ (in cs). For $y \in \omega$ we write $y^\beta$ instead of $\{y\}^\beta$.

$X^\beta$ is a sequence space with $\varphi < X^\beta < \omega$

$X \subset Y \subset \omega \Rightarrow Y^\beta < X^\beta$

$X \subset X^\beta \beta := (X^\beta)^\beta$

$\varphi^\beta = \omega$ and $\omega^\beta = \varphi$

$X \subset \omega_A \Leftrightarrow \forall n \in \mathbb{N}^0 \forall x = (x_k) \in X : \sum_k a_{nk}x_k$ converges $\Leftrightarrow \forall n \in \mathbb{N}^0 : (a_{nk})_k \in X^\beta$

**Theorem.** (2.3.3)

(a) $c_0^\beta = c^\beta = m^\beta = \ell$
Corollary. (2.3.4) Let $A = (a_{nk})$ be any infinite matrix. Then

\[ \sum_k |a_{nk}| < \infty, \text{ that is } (a_{nk})_k \in \ell \text{ for each } n \in \mathbb{N}^0 \]

if and only if one (thus each) of the inclusions $m \subset \omega_A$, $c \subset \omega_A$, $c_0 \subset \omega_A$, $m_0 \subset \omega_A$, and $\tau \subset \omega_A$ holds.

\[ m_A := \{ x \in \omega_A | Ax \in m \} \]

Theorem (c $\subset m_A$). (2.3.5) For any matrix $A = (a_{nk})$ the following statements are equivalent:

(a) $m \subset m_A$, that is $m \subset \omega_A$ and $A(m) \subset m$,

(b) $c \subset m_A$, that is $c \subset \omega_A$ and $A(c) \subset m$,

(c) $c_0 \subset m_A$, that is $c \subset \omega_A$ and $A(c_0) \subset m$,

(d) $||A|| := \sup_n \sum_k |a_{nk}| < +\infty$.

Although this result can be shown easily using uniform boundedness principle, the proof used in this book illustrates “gliding hump” argument. This type of arguments is often used in functional analysis and also in connection with matrix methods [S]. (In Theorem 7.4.7 the proof of the same fact is given, using UBP.)

Theorem (conservative, regular for null sequences). (2.3.6) Let $A = (a_{nk})$ be an infinite matrix.

I. The following statements are equivalent:

(a) $A$ is conservative for null sequences (that is, $c_0 \subset c_A$).

(b) $c_0 \subset \omega_A$ and $A(c_0) \subset c$.

(c) $A$ satisfies

(Zn) $||A|| < \infty$ (row norm condition) and

(Sp) $\forall k \in \mathbb{N}^0$: $a_k = \lim_n a_{nk}$ exists (column condition).

LIMIT FORMULA: If $A$ is conservative for null sequences then

\[ (a_k) \in \ell \quad \text{and} \quad \lim_A x = \sum_k a_k x_k \quad (x = (x_k) \in c_0). \]

II. The following statements are equivalent:

(a) $A$ is regular for null sequences (that is, $c_0 \subset c_A$ and $\lim_A |c_0 = 0$).
(b) $c_0 \subset \omega_A$ and $A(c_0) \subset c_0$.

(c) $A$ satisfies (Zn) with $a_k = 0$ for all $k \in \mathbb{N}^0$, that is (Sp0) $\forall k \in \mathbb{N}^0$: $(a_{nk})_k \in c_0$.

**Theorem (of Toeplitz, Silverman, Kojima and Schur).** (2.3.7) Let $A = (a_{nk})$ be an infinite matrix.

I. The following statements are equivalent:

(a) $A$ is conservative (that is, $c \subset c_A$).

(b) $c \subset \omega_A$ and $A(c) \subset c$.

(c) $A$ satisfies (Zn), (Sp) and

$$(Zs) \sum_k a_{nk} \ (n \in \mathbb{N}^0)$$

and $a := \lim_n \sum_k a_{nk}$ exist (row sum condition).

**LIMIT FORMULA:** If $A$ is conservative, then

$$\lim_A x = \chi(A) \lim x + \sum_k a_k x_k \quad (x = (x_k) \in c),$$

where

$$\chi(A) := \lim_A e - \sum_k \lim_A e^k = \lim_n \sum_k a_{nk} - \sum_k \lim a_{nk} = a - \sum_k a_k$$

is called the characteristic of $A$.

II. The following statements are equivalent:

(a) $A$ is regular (that is, $c \subset c_A$ and $\lim_A |e|_{c} = \lim$).

(b) $A$ is regular for null sequences and $e \in c_A$ with $\lim_A e = 1$.

(c) $A$ satisfies (Zn), (Sp0) and the condition $(Zs_1)$, that is $(Zs)$ with $a = 1$.

Moreover, if $A$ is regular, then $\chi(A) = 1$.

**Theorem.** (2.3.8) If $A$ is a matrix which sums all thin sequences, then $A$ is conservative for null sequences, that is $\mathcal{T} \subset c_A$ implies $c_0 \subset c_A$ and, in particular, $\|A\| < \infty$. Moreover, $A$ is conservative, if $\mathcal{T} \cup \{e\} \subset c_A$.

**Definition.** (2.3.9) A conservative matrix $A$, and the corresponding matrix method, is called coregular if $\chi(A) \neq 0$ and conull if $\chi(A) = 0$.

Bibliography: [H, K, P, W]

### 2.4 Coercive and strongly conservative matrix methods

**Theorem (Schur).** (2.4.1) For a matrix $A = (a_{nk})$ the following statements are equivalent:

(a) $A$ is coercive, that is $m \subset c_A$. 

(b) $A$ satisfies $(Sp)$, and $\sum_k |a_{nk}|$ converge uniformly for $n \in \mathbb{N}^0$.

(c) $c_0 \subset c_A$ and $h(A) := \limsup_n \sum_k |a_{nk} - a_k| = 0$.

Corollary. (2.4.2) Every coercive matrix is conull. In particular, a matrix cannot be both regular and coercive.

Theorem (Hahn). (2.4.5) If a matrix $A$ sums all sequences of zeros and ones, then it sums all bounded sequences. That is, $\chi \subset c_A$ implies $m \subset c_A$.

Definition (strong regularity). (2.4.6) A summability method $V = (V, N_V, V - \lim)$ is called strongly regular if $V$ is strongly conservative (that is, $f \subset N_V$) and $V - \lim F = F - \lim$. A matrix is called strongly regular if the corresponding matrix method is strongly regular.

Theorem (Lorentz). (2.4.9) Let $A = (a_{nk})$ be a conservative matrix and $a_k$ be the limit of the $k^{th}$ column of $A$. Then the following statements are equivalent:

(a) $A$ is strongly conservative.

(b) $\limsup_n \sum_k |a_{nk} - a_{n,k+1} - a_k + a_{k+1}| = 0$.

LIMIT FORMULA: If $A$ is strongly conservative, then

$$\lim_{A} x = \chi(A) \cdot F - \lim x + \sum_k a_k x_k \quad (x = (x_k) \in f).$$

$$f_0 := \{x \in f; F - \lim x = 0\}$$

Lemma. $bs \subset f_0$

Theorem. (2.4.12) If $A$ is any matrix then the following statements are equivalent:

(a) $A$ is strongly regular.

(b) $A$ is regular and strongly conservative.

(c) $A$ is regular and satisfies $\limsup_n \sum_k |a_{n,k} - a_{n,k+1}| = 0$.

Example. (2.4.13) The discrete Borel method $B_1^\ast$ is strongly regular.\textsuperscript{5}

Bibliography: [L, P, R, W, ZB]

\textsuperscript{4}Note that $h(A)$ is defined for each matrix $A$ being conservative for null sequences.

\textsuperscript{5}The proof of [B] claims $e^{-n} \sum_{k=n}^{\infty} \frac{a_k}{k^n}$ converges to 0. This is not correct. TODO Try to find a correct proof.
2.5 Abundance within domains; factor sequences

Definition (factor sequence). (2.5.1) Let $A$ be a matrix, let $x = (x_k) \in c_A$ and $y = (y_k) \in \omega$. Then $y$ is called a factor sequence for $x$ and $A$, if $yx = (y_kx_k) \in c_A$.

Theorem. (2.5.3) Let $A = (a_{nk})$ be a matrix which satisfies $(Sp)$. Let $a_k$ denote the limit of the $k^{th}$ column of $A$ and let $x = (x_k) \in c_A$ be given such that $\sum_k a_k x_k$ converges. If
\[
\lim_{A} x = \sum_k a_k x_k \quad (2)
\]
and
\[
\sup_{\nu, n} \left\| \sum_{k=0}^{\nu} a_{nk} x_k \right\| < \infty \quad (3)
\]
the there exists an index sequence $(r_j)$ such that each sequence $y = (y_k)$ which satisfies
\[
\sum_{k=r_j+1}^{r_{j+1}} |y_k - y_{k-1}| \to 0 \quad (j \to \infty) \quad (\mathcal{O})
\]
is a factor sequence for $x$ and $A$. Moreover,
\[
\sum_k a_k y_k x_k \text{ converges and } \lim_{A} yx = \sum_k a_k y_k x_k.
\]

A sequence which satisfies the condition $(\mathcal{O} \mathcal{I})$ for some index sequence $r = (r_j)$ is called slowly oscillating (with respect to $r$).

Definition. (2.5.4) More generally than in 2.3.7I we define
\[
\chi(G) := \lim_n \sum_k g_{nk} - \sum_k \lim_n g_{nk} = g - \sum k g_k
\]
for any matrix $G = (g_{nk})$ that satisfies the conditions $(ZS)$ and $(Sp)$ and for which $\sum k g_k$ converges.
2.6 Comparison and consistency theorems
2.7 Triangles of type M
3 Special summability methods
4 Tauberian theorems
5 Application of boundary methods

Part II: Functional analytic methods in summability

6 Functional analytic basis
6.1 Topological spaces
6.2 Semi-metric spaces
6.3 Semi-normed spaces, Banach spaces
6.4 Locally convex spaces
Example. (6.4.15) TODO (definition of $\tau_\omega$)

6.5 Continuous linear maps and the dual space of a locally convex spaces
6.6 Dual pairs and compatible topologies
6.7 Fréchet spaces

A complete metrizable locally convex spaces is called a Fréchet space, or an $F$-space.

6My note: Also the book [M] deals with summability, e.g. in Section 2.4 Application of Basic Principles.
7 Topological sequences spaces: K- and FK-spaces

7.1 Sequence spaces and their ξ-duals

7.2 K-spaces

K-spaces = locally convex topologies that are stronger than τω. In such stronger topologies, convergence implies coordinatewise convergence.

Definition (K-space). (7.2.2) A locally convex space (X, τ) is called a K-space if X < ω and τω ⊂ τ. In such a case τ is called a K-topology on X.

7.3 FK-spaces

Definition (FK- and BK-space). A locally convex space (X, τ) is called an FK-space and τ is called an FK-topology if (X, τ) is both, a K-space and an F-space. By definition, a BK-space is a normable FK-space and its topology is called BK-topology.

Example (7.3.2). (a) (ω, τω) is an FK-space but no BK-space
(b) m, c, c0, and f endowed with ∥·∥∞ are BK-spaces.

Definition (matrix map). (7.3.6) Let X and Y be a sequence spaces over K, and let T: X → Y be a linear map. Then T is called a matrix map if there exists a matrix A = (a_nk) such that X ⊂ ω_A and T(x) = Ax for all x ∈ X. (Where no confusion can arise, we denote both the matrix map and the matrix by the same letter.)

Corollary. (7.3.7) Matrix maps between FK-spaces are continuous.7

Theorem (subspace). (7.3.8) Every closed subspace of an FK-space (endowed with the subspace topology) is an FK-space.

7.4 Functional analytic proofs of some Toeplitz-Silverman-type theorems

Theorem (c ⊂ m_A). (7.4.2=2.3.5) For any matrix A = (a_nk) the following statements are equivalent
(a) m ⊂ m_A, that is m ⊂ ω_A and A(m) ⊂ m,
(b) c ⊂ m_A, that is c ⊂ ω_A and A(c) ⊂ m,
(c) c0 ⊂ m_A, that is c ⊂ ω_A and A(c0) ⊂ m,

7My note: Summation matrix Σ defines a matrix map ℓ → m or cs → m. If I take sup-norm on the domain, it is not continuous. However, the subspaces ℓ and cs are not closed in the sup-norm; just note that if x(k) is defined by x(k)_n = \begin{cases} \frac{1}{n}; & n \leq k \\ \frac{1}{n^2}; & n > k \end{cases} then x(k) → (\frac{1}{k}) in m.
(d) \(\|A\| := \sup_n \sum_k |a_{nk}| < +\infty.\)

The authors provide two proofs. The first one is based on the uniform boundedness principle and the second one on Theorem 7.3.7. They also show \(\|A\| = \|A\|_{X,m}\) for \(X \in \{c_0, c, m\}\).

8 Matrix methods: structure of the domains

Part III: Combining classical and functional analytic methods

9 Consistency of matrix methods

10 Saks spaces and bounded domains

11 Some aspect of topological sequence spaces

11.1 An inclusion theorem

11.2 Gliding hump and oscillating properties

11.3 Theorems of Toeplitz-Silverman type via sectional convergence and

11.4 Barelled K-spaces

11.5 The sequences of zeros and ones in a sequences spaces

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*TODO Check in the book where they define \(\|A\|_{X,Y}\). I think it is the norm of Theorem 6.3.19, although they do not use this notation in that theorem.
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