[B, p.59, Example 2.4.13c]

$$\lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} \frac{n^k}{k!}}{e^n} = 0$$

Stirling: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\sum_{k=n}^{\infty} \frac{n^k}{k!} = \frac{n^n}{n!}\left(1+\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\dots\right)$$

First attempt:

$$\sum_{k=n}^{\infty} \frac{n^k}{k!} = \frac{n^n}{n!}\left(1+\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\dots\right) \le$$
$$\frac{n^n}{n!}\left(1+\frac{n}{n+1}+\frac{n^2}{(n+1)^2}+\dots\right) = \frac{n^n}{n!}(n+1)$$

Using Stirling Formula we see that the RHS is asymptotically $e^n \frac{n+1}{\sqrt{2\pi n}}$, which does not help to show the desired result.

Disproof? For any positive integer k we have

$$1+\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\dots+\frac{n^k}{(n+1)\dots(n+k)} \ge$$
$$\ge 1+\frac{n}{n+k}+\frac{n^2}{(n+k)^2}+\dots+\frac{n^k}{(n+k)^k} = \frac{1-\frac{n^{k+1}}{(n+k)^{k+1}}}{1-\frac{n}{n+k}} = k\left(1-\frac{n^{k+1}}{(n+k)^{k+1}}\right)$$

When choosing $k=\lfloor\sqrt{n}\rfloor-1$ we get the RHS approximately $\sqrt{n}\left(1-\frac{n^{\sqrt{n}}}{(n+\sqrt{n})^{\sqrt{n}}}\right)$. Since

$$\frac{n^{\sqrt{n}}}{(n+\sqrt{n})^{\sqrt{n}}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} \longrightarrow \frac{1}{e},$$

we get $1-\frac{n^{\sqrt{n}}}{(n+\sqrt{n})^{\sqrt{n}}} \ge \frac{1}{2}$ for n large enough.

Thus (for large n) we get

$$\sum_{k=n}^{\infty} \frac{n^k}{k!} = \frac{n^n}{n!}\left(1+\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\dots\right) \ge \frac{n^n}{n!}\frac{\sqrt{n}}{2} \sim \frac{e^n}{2\sqrt{2\pi}}$$

This yields the lower bound $\frac{1}{2\sqrt{2\pi}}$ for the limit in question.

References

[B] J. Boos. *Classical and modern methods in summability.* Oxford University Press, New York, 2000.