1. Is the norm of an infinite matrix A the same as the norm of the induced linear function $\ell_\infty \to \ell_\infty$?

Yes.

$$\|A\| := \sup_n \sum_k |a_{nk}|$$

We have a sequence (n_i) such that $\lim\limits_{i\to\infty} \sum_k |a|_{n_i k} = \|A\|$. By putting $x^{(i)} = (\operatorname{sgn} a_{n_i k})_k$ we construct a sequence of elements of the unit closed ball in ℓ_∞ such that $Ax^{(i)} = \sum_k |a|_{n_i k}$, i.e., $\lim\limits_{i\to\infty} Ax^{(i)} = \|A\|$.

Thus the norm of the induced linear function is at least $\|A\|$. The opposite inequality follows from the estimate

$$\|Ax\|_\infty \le \|A\|\|x\|_\infty$$

This is very similar to the way a matrix norm is obtained from a given norm vectors in the finite-dimensional spaces.

[WIK, Matrix norm]: If vector norms on K^m and K^n are given (K is field of real or complex numbers), then one defines the corresponding induced norm or operator norm on the space of m-by-n matrices as the following maxima:

$$\begin{aligned}
\|A\| &= \max\{\|Ax\| : x \in K^n \text{ with } \|x\| \le 1\} && (1)\\
&= \max\{\|Ax\| : x \in K^n \text{ with } \|x\| = 1\} && (2)\\
&= \max\left\{\frac{\|Ax\|}{\|x\|} : x \in K^n \text{ with } x \ne 0\right\}. && (3)
\end{aligned}$$

In the case of $p = 1$ and $p = \infty$, the norms can be computed as:

$$\begin{aligned}
\|A\|_1 &= \max_{1\le j\le n} \sum_{i=1}^m |a_{ij}| && (4)\\
\|A\|_\infty &= \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|. && (5)
\end{aligned}$$

2. Can we give a constructive example of an unbounded linear function $\ell_\infty \to \ell_\infty$?

An unbounded linear function can be constructed using Hamel bases, but here we would like to find a choice-free example. As shown bellow, we need at least some form of AC. (It is consistent with ZF that there is no unbounded linear function.)

In this connection also the following result is worth mentioning: [B, Corollary 7.3.7]: Matrix maps between FK-spaces are continuous.

Since ℓ_∞ is an FK-space, this result implies that every linear function given by an infinite matrix, such that it maps bounded sequences on bounded sequences, must be continuous.

Of course, if we do not insist that the domain of the linear map should be the whole ℓ_∞, we can easily find examples of unbounded linear functions. E.g. the summation matrix $\Sigma := \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ (corresponding to the map which assigns to a sequence the sequence of the partial sums) is an unbounded linear function from ℓ_1 to ℓ_∞. (Note that ℓ_1, considered as the subspace of ℓ_∞, is not closed, hence not complete.)

[HR, Note 96]: X is a Fréchet space if X is locally convex and the topology on X is induced by complete invariant metric.

In the model $\mathcal{M}5$ (Solovay [S]) form 43 (DC) and the negation of form 280 hold (the negation of 280 is "Every subset of a complete, separable metric space has the Baire property.")[1] Wright [W] shows that the following are consistent by showing that they are theorems of ZF+DC+¬form 280:

A. If X and Y are separable metrizable topological groups and X is complete and $H\colon X \to Y$ is a group homomorphism then H is continuous.

B. If X is a Fréchet space and Y is a separable, metrizable topological vector space and $T\colon X \to Y$ is linear, then T is continuous.

C. Every linear functional on a Fréchet space is continuous.

D. If X and Y are Fréchet spaces, $T\colon X \to Y$ is linear and there are enough linear functionals on Y to separate the points of Y then T is continuous.

E. If H is a Hilbert space and $T\colon H \to H$ is linear then T is bounded.

References

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[S] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics*, 92:1–56, 1970.

[W] J. D. M. Wright. All operators on a Hilbert space are bounded. *Bull. Amer. Math. Soc.*, 79(6):1247–1250, 1973.

[WIK] Wikipedia. `http://en.wikipedia.org`.

[1] A set $Y \subset X$ of a topological space X has the *Baire property* if $(Y \setminus U) \cup (U \setminus Y)$ is meager (=of the first category) for some open set U.