1 Equivalence of $\mathcal{I}$-continuity and continuity at a point for real functions

Main result is Theorem 1 which answers the question posed on the last seminar. Throughout the paper we assume that $\mathcal{I}$ is an admissible ideal.

First we recall some basic notions from general topology (see e.g. [2]). Let X, Y be topological spaces. Then a map $f\colon X \to Y$ is continuous at a point $x \in X$ if for every neighborhood $V \subseteq Y$ of $f(x)$ there exists a neighborhood $U \subseteq X$ of x such that $f[U] \subseteq V$. A map $f\colon X \to Y$ is continuous if and only if it is continuous at each point of X.

We next recall some definitions from [4]. They are very straightforward generalizations of corresponding notions defined for real functions in [1].

Definition 1. Let $\mathcal{I}$ be an ideal in $\mathbb{N}$. A sequence $(x_n)_{n=1}^{\infty}$ in a topological space X is said to be $\mathcal{I}$-*convergent* to a point $x \in X$ if

$$A(U) = \{n : x_n \notin U\} \in \mathcal{I}$$

holds for each open neighborhood U of x. We denote it by $\mathcal{I}\text{-}\lim x_n = x$.

If $\mathcal{I} = \mathcal{I}_f$ is the Fréchet ideal (i.e. the ideal containing exactly all finite subsets of $\mathbb{N}$) then $\mathcal{I}$-convergence coincides with the usual convergence.

Definition 2. Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ and X, Y be topological spaces and $x_0 \in X$. A map $f\colon X \to Y$ is called $\mathcal{I}$-*continuous at* x_0 if for each sequence $(x_n)_{n=1}^{\infty}$ in X

$$\mathcal{I}\text{-}\lim x_n = x_0 \quad \Rightarrow \quad \mathcal{I}\text{-}\lim f(x_n) = f(x_0)$$

holds.

Given a topological space X and a point $a \in X$, denote by X_a the space constructed by making each point, other then a, isolated with a retaining its original neighborhoods. (I.e. a subset $U \subseteq X$ is open in X_a if and only if $a \notin U$ or there exists an open subset V of X such that $a \in V \subseteq U$.) The topological space X_a is called *prime factor of* X *at the point* a.

In what follows we examine the relationship between $\mathcal{I}$-continuity of a function $f\colon X \to Y$ at a point $a \in X$ and $\mathcal{I}$-continuity of the function $f\colon X_a \to Y$ for topological spaces X, Y.

Lemma 1. *Let X be a topological space, $a \in X$ and $(x_n)_{n=1}^{\infty}$ be a sequence of points of X. Then $\mathcal{I}\text{-}\lim x_n = a$ in X if and only if $\mathcal{I}\text{-}\lim x_n = a$ in X_a.*

Proof. This is an easy consequence of the fact that neighborhoods of a are the same in both topological spaces X and X_a, so in the definition of $\mathcal{I}$-limit we have precisely the same system of sets $A(U)$ for both spaces. $\square$

By letting $\mathcal{I}_f = \mathcal{I}$ we get:

Corollary 1. *Let X be a topological space, $a \in X$ and $(x_n)_{n=1}^{\infty}$ be a sequence of points of X. Then $\lim x_n = a$ in X if and only if $\lim x_n = a$ in X_a.*

Lemma 2. *If $x \in X$ is an isolated point of a topological space X (i.e. the set $\{x\}$ is open in X) then any map $f\colon X \to Y$ is $\mathcal{I}$-continuous at x.*

Dôkaz. If $U = \{x\}$ then $A(U) = \{n : x_n \neq x\} \in \mathcal{I}$. Therefore for each $V \subset Y$ such that $f(x) \in V$ we get $A(V) = \{n : f(x_n) \notin V\} \subset \{n : f(x_n) \neq f(x)\} \subset \{n : x_n \neq x\}$ and V belongs to $\mathcal{I}$. Hence $\mathcal{I}$-$\lim f(x_n) = f(x)$. □

Proposition 1. *Let X, Y be topological spaces and $a \in X$. Then $f\colon X \to Y$ is $\mathcal{I}$-continuous at a if and only if $f\colon X_a \to Y$ is $\mathcal{I}$-continuous.*

Proof. The map $f\colon X_a \to Y$ is obviously $\mathcal{I}$-continuous at each point different from a, since these points are isolated (Lemma 2). If follows from Lemma 1 that continuity of $f\colon X \to Y$ at a is equivalent to continuity of $f\colon X_a \to Y$ at a. □

If $X = \mathbb{R}$ then a map $f\colon X \to Y$ is continuous if and only if it is $\mathcal{I}_f$-continuous. This implies

Corollary 2. *Let Y be a topological spaces. A map $f\colon \mathbb{R} \to Y$ is continuous at a if and only if it $f\colon \mathbb{R}_a \to Y$ is continuous.*

We recall notions of sequential and Fréchet spaces (see e.g. [2], a thorough survey of these types of spaces and some related properties and problems is e.g. paper [3] available at `http://xyz.lanl.gov/abs/math.GN/0412558`). A topological space X is called a *sequential space* if a set $A \subseteq X$ is closed if and only if together with any sequence it contains all its limits. A topological space X is called *Fréchet space* if for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence $(x_n)_{n=1}^{\infty}$ of points of A converging to X. Every first-countable (in particular every metric space) is a Fréchet space and every Fréchet space is a sequential space.

Proposition 2. *The space $\mathbb{R}_a$ for any $a \in \mathbb{R}$ is Fréchet.*

Dôkaz. We need to show that if $x \in \overline{V}$ then there is a sequence of points of V converging to x. If x is an isolated point then $x \in \overline{V}$ implies $x \in V$ and we can use the constant sequence $x_n = x$. So it remains to solve the case $x = a$.

Let $V \subseteq \mathbb{R}$ and $a \in \overline{V}$ in $\mathbb{R}_a$. This means that each neighborhood of a contains a point from V. Neighborhoods of a in the topological space $\mathbb{R}$ are the same as in the topological space $\mathbb{R}_a$, therefore $a \in \overline{V}$ in $\mathbb{R}$. Since $\mathbb{R}$ is a Fréchet space, there exists a sequence $(x_n)_{n=1}^{\infty}$ of points of V which converges to a in the space $\mathbb{R}$. Using Corollary 1 we get that it converges to a in $\mathbb{R}_a$ as well. □

Proposition 3 ([4, Corollary 2.4]). *Let X be a sequential space and let $\mathcal{I}$ be an admissible ideal. Let Y be a topological space and let $f\colon X \to Y$ be a map. Then the following statements are equivalent:*

(1) f is continuous,

(2) f is $\mathcal{I}_f$-continuous,

(3) f is $\mathcal{I}$-continuous.

Theorem 1. *Let $a \in \mathbb{R}$. A map $f\colon \mathbb{R} \to Y$ is continuous at a if and only if $f\colon \mathbb{R} \to Y$ is $\mathcal{I}$-continuous at a.*

Sketch of the proof. $f\colon \mathbb{R} \to Y$ is continuous at $a \Leftrightarrow$ (Corollary 2) $f\colon \mathbb{R}_a \to Y$ is continuous $\Leftrightarrow$ (Proposition 2 and Proposition 3) $f\colon \mathbb{R}_a \to Y$ is $\mathcal{I}$-continuous $\Leftrightarrow$ (Proposition 1) $f\colon \mathbb{R} \to Y$ is continuous at a. □

Proposition 1 can be strengthened without any changes in the proof.

Proposition 4. *If X is a Fréchet space then each prime factor X_a of X is a Fréchet space*

Also the claim of Corollary 1 holds for any topological space X. This follows again from the fact that a has the same neighborhoods in both topological spaces X and X_a.

Proposition 5. *Let X, Y be topological spaces and $a \in X$. A map $f\colon X \to Y$ is continuous at a if and only if $f\colon X_a \to Y$ is continuous.*

Using these fact we arrive to conclusion that $\mathcal{I}$-continuity, $\mathcal{I}_f$-continuity and continuity at a point $a \in X$ are equivalent for any Fréchet space X. It is natural to ask if the same holds for any sequential space.

References

[1] V. Baláž, J. Červeňanský, P. Kostyrko, and T. Šalát. I-convergence and I-continuity of real functions. *Acta Mathematica, Faculty of Natural Sciences, Constantine the Philosopher University Nitra*, 5:43–50, 2002.

[2] R. Engelking. *General Topology*. PWN, Warsaw, 1977.

[3] A. Goreham. Sequential convergence in topological spaces. 2004. arXiv:math.GN/0204116.

[4] M. Sleziak. I-continuity in topological spaces. *Acta Mathematica, Faculty of Natural Sciences, Constantine the Philosopher University Nitra*, 6:115–122, 2003.