$\mathcal{F}$-convergence, filters and nets

The main purpose of these notes is to compare several notions that describe convergence in topological spaces.

We are most interested in $\mathcal{F}$-convergence and we want to show how it relates to other types of convergence which are usually studied in general topology. (In fact, the only books I know, in which convergence is developed using $\mathcal{F}$-convergence are [Bo, Di].)

We assume that the reader is familiar with convergence of filters and nets in topological spaces. (Although we include definitions and overview of basic results.) It would be possible to make a unified approach to these two notions by introducing $\mathcal{F}$-convergence first and then treating nets and filters as a special cases. This is, however, not the goal of this text; we would have organized it quite differently if we chose this approach. Still for some of the results we mention the reader can choose either the proof relying on the corresponding results on convergence of filters or he can read the self-contained proof for $\mathcal{F}$-convergence, from which he can obtain the result for convergence of filters as a special case. (In such situations, we included usually both proofs.)

We tried to include detailed proofs (even for the claims that may be considered easy).

We will show that convergence of nets is a special case of $\mathcal{F}$-convergence, the same is true for convergence of filters. On the other hand, $\mathcal{F}$-convergence can be expressed using the convergence of filters (Theorem 2.2). The same is true for many other notions related to filters and nets (cluster points, limit superior, Cauchy nets and filters). However, we do not claim that any of these notions is redundant or not useful. Each of them brings different insights, different possibilities for generalization and can be useful in some situations.

1 Preliminaries

1.1 Basic definitions

1.1.1 Filters

Definition 1.1. A filter on a set M is a non-empty family $\mathcal{F} \subseteq \mathcal{P}(M)$ such that:

(i) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(ii) $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$;

(iii) $\emptyset \notin \mathcal{F}$.

A filter is called *free* if $\bigcap \mathcal{F} = \emptyset$.

{REMPROPER}

Remark 1.2. Some authors allow $\emptyset \in \mathcal{F}$ in the definition of filter. (They use the term *proper filter* for our definition.)

An important example of a filter is *Fréchet filter* $\mathcal{F}_0 = \{A \subseteq \mathbb{N}; \mathbb{N} \setminus A \text{ is finite}\}$ on the set $\mathbb{N}$. Similarly, we can define Fréchet filter on any set as $\mathcal{F}_0(M) = \{A \subseteq M; M \setminus A \text{ is finite}\}$. Note that a filter $\mathcal{F}$ on a set M is free if and only if $\mathcal{F}_0(M) \subseteq \mathcal{F}$.

Definition 1.3. A filter $\mathcal{F}$ on M is called *ultrafilter* if for every $A \subseteq M$ either $A \in \mathcal{F}$ or $M \setminus A \in \mathcal{F}$.

Ultrafilters are precisely the maximal filters with respect to inclusion. Using AC it can be shown that every system of sets, which has finite intersection property, is contained in an ultrafilter.

Definition 1.4. A system $\mathcal{B} \subseteq \mathcal{P}(S)$ is called *filterbase* if
(i) $\mathcal{B} \neq \emptyset$,
(ii) $A, B \in \mathcal{B} \Rightarrow (\exists C \in \mathcal{B})\ C \subseteq A \cap B$.
If $\mathcal{B}$ is a filterbase, then the system

$$\mathcal{F} = \{A \supseteq B; B \in \mathcal{B}\}$$

is a filter. It is called filter *generated* by the base $\mathcal{B}$.

1.1.2 $\mathcal{F}$-limit

{DEFFLIM}

Definition 1.5. Let $\mathcal{F}$ be a filter on a set M and X be a topological space. A function $f\colon M \to X$ is said to be *$\mathcal{F}$-convergent to $x \in X$,* or that x is *$\mathcal{F}$-limit* of the function f, if

$$f^{-1}(U) = \{s \in S; f(s) \in U\} \in \mathcal{F}$$

holds for every neighborhood U of the point x.

We use the notation

$$\mathcal{F}\text{-}\lim f = x.$$

The definition of $\mathcal{F}$-limit can be rewritten as

$$(\forall U \in \mathcal{N}(x)) f^{-1}(U) \in \mathcal{F}$$

(where $\mathcal{N}(x)$ denotes the *neighborhood filter* of x – the set of all neighborhoods of x).

For the Fréchet filter $\mathcal{F}_0$ we obtain the usual convergence of sequences. For the neighborhood filter $\mathcal{N}(m)$ of a point $m \in M$ we get precisely the usual notion of limit of a function f at the point m. (Of course, we have to assume that M is a topological space in this second example.)

Many further examples of $\mathcal{F}$-convergence for filters on $\mathbb{N}$ are given in [KŠW, KMŠS].

The definition of limit with respect to a filter dates back to Henri Cartan [Bo, p.70–71].

Let us note, that several authors worked with ideals instead of filters, see [KŠW]. The paper [KŠW] defines also the notion of $\mathcal{I}^*$-convergence, which we will briefly mention later. (We will call it $\mathcal{F}^*$-convergence, since we work with filters instead of ideals.) For more details on he the history of the notions of $\mathcal{F}$-convergence and $\mathcal{F}^*$-convergence we refer the reader to the paper [MS].

{REMNOTHAUS}

Remark 1.6. In general, $\mathcal{F}$-limit of a function need not be determined uniquely. Therefore the notation $x \in \mathcal{F}\text{-}\lim f$ would be more precise than our notation. However, $\mathcal{F}$-limits are mostly applied to the situation where the topological space X is Hausdorff, so our notation should not be confusing. (It is also closer to the usual notation for limits of real functions and real sequences.)

We will occasionally use also the notation $\mathcal{F}\text{-}\lim f$ also for the set of all $\mathcal{F}$-limits. (Similarly for other types of limits we will be speaking about.)

1.1.3 Convergence of nets in topological spaces

Definition 1.7. We say that $(D, \leq)$ is a *directed set,* if $\leq$ is a relation on D such that

(i) $x \leq y \wedge y \leq z \Rightarrow x \leq z$ for each $x, y, z \in D$;

(ii) $x \leq x$ for each $x \in D$;

(iii) for each $x, y \in D$ there exist $z \in D$ with $x \leq z$ and $y \leq z$.

In the other words a directed set is a set with a relation which is reflexive, transitive (=preorder or quasi-order) and upwards-directed.

The following two notions will be often useful for us

Definition 1.8. A subset A of set D directed by $\leq$ is *cofinal* in D if for every $d \in D$ there exists an $a \in A$ such that $d \leq a$.

A subset A of a directed set D is called *residual* if there is some $d_0 \in D$ such that $d \geq d_0$ implies $d \in A$.

Definition 1.9. A *net* in a topological space X is a map from any non-empty directed set D to X. It is denoted by $(x_d)_{d \in D}$.

The convergence of nets is defined analogously to the usual notion of convergence of sequences.

{DEFLIMNET}

Definition 1.10. Let $(x_d)_{d \in D}$ be a net in a topological space X is said to be *convergent* to $x \in X$ if for each neighborhood U of x there exists $d_0 \in D$ such that $x_d \in U$ for each $d \geq d_0$.

$$(\forall U \in \mathcal{N}(x))(\exists d_0 \in D)(\forall d \geq d_0) x_d \in U \tag{1}$$ {EQ1}

If a net $(x_d)_{d \in D}$ converges to x, the point x is called a *limit* of this net.

The set of all limits of a net is denoted $\lim x_d$.

The property (1) characterizing the convergence of net is sometimes called x_d *is eventually in* U or *residually in* U, e.g. [K, P, W]. In the other words, the set of all d's with $x_d \in U$ is residual in D.

1.1.4 Convergence of filters in topological spaces

Definition 1.11. A filter on a topological space X is said to be *convergent* to a point $x \in X$ (or x is called a *limit* of $\mathcal{F}$) is every neighborhood of x belongs to $\mathcal{F}$. The set of all limits of $\mathcal{F}$ is denoted by $\lim \mathcal{F}$.

A filterbase $\mathcal{B}$ on X is said to converge to x if the filter generated by this filterbase converges to x.

We see that $\mathcal{F} \to x \Leftrightarrow \mathcal{N}(x) \subseteq \mathcal{F}$.

{REMFILTSPECCASE}

Remark 1.12. Observe that this is a special case of $\mathcal{F}$-convergence: for $M = X$ and $f = id_X$. (We will explain bellow that convergence of nets can also be considered as a special case of $\mathcal{F}$-convergence.)

1.2 Some known facts about nets and filters

In metric spaces, the topology is uniquely determined by the convergent sequences. This is no longer true for arbitrary topological spaces. The notions of nets and filters can both replace sequences to obtain analogous results.

Both approaches have their merits. The definition of limit of a net is very similar to the definition of a limit of a sequence. Therefore the intuition obtained when working with metric spaces can be useful when working with nets. However, the definition of a subnet is quite delicate, whereas the counterpart − a finer filter − is a very simple notion.

Introduction to the convergence of nets and filters can be found in many textbooks, not only in the area of general topology. (Although some authors include only one of these notions. Despite the fact that there is a correspondence between nets and filters and results can be translated between these two languages, I believe that it is useful to know about both of them.) Very good introduction, containing many historical details and comparing these two approaches in depth, is the text [Cl].

1.2.1 Results on convergence of nets

The following theorems show that the convergence of nets describes completely the topology of X (and, consequently, it also characterizes continuity).

Theorem 1.13. *A point x belongs to $\overline{A}$ if and only if there exists a net consisting of elements of A which converges to x.*

{THMNETCLOSED}

Theorem 1.14. *A subset V of a topological space X is closed if and only if for each net $(x_d)_{d\in D}$ such that $x_d \in V$ for each $d \in D$, every limit of $(x_d)_{d\in D}$ belongs to V as well.*

{THMNETCONT}

Theorem 1.15. *Let X, Y be topological spaces. A map $f\colon X \to Y$ is continuous if and only if, whenever a net x_d converges to x, the net $f(x_d)$ converges to $f(x)$.*

Several important notions, such as Hausdorffness and compactness, can be characterized with the help of nets.

{THMNETHAUS}

Theorem 1.16. *A topological space X Hausdorff $\Leftrightarrow$ every net in X has at most one limit.*

In these notes we will not discuss the notion of subnet. The reason is that, compared to the notion of subsequence, the "correct" notion of subnet is more complicated. (Since the counterpart to the notion of subnet is very simple notion of finer filter, this is one of the points in which filters seem to be easier to deal with than nets.)

Let us just mention that three different notions of subnet appear in literature, all of them work well in the sense that the theorems about subnets which we mention here are true for each of these definitions. These three definitions of subnet are compared e.g. in [S, 7.14–7.20]. They are also briefly discussed in [Cl].[1]

{THMNETCOMPSUBNET}

Theorem 1.17. *A topological space is compact if and only if every net in X has a convergent subnet.*

1.2.2 Results on convergence of filters

The following theorem shows that convergence of filters determines which subsets of a topological space are closed, hence it completely describes the topology.

{THMFILTCLOS}

Theorem 1.18. *The point x belongs to $\overline{A}$ if and only if there exists a filterbase consisting of subsets of A converging to x.*

{THMFILTCONT}

Theorem 1.19. *A mapping f of a topological space X to a topological space Y is continuous if and only if for every filter-base $\mathcal{G}$ in the space X and the filter-base $f(\mathcal{G}) = \{f[A] : A \in \mathcal{G}\}$ in the space Y we have*

$$f[\lim \mathcal{G}] \subset \lim f(\mathcal{G}).$$

{THMFILTHAUS}

Theorem 1.20. *A topological space X Hausdorff $\Leftrightarrow$ every filter in X has at most one limit.*

{THMFILTCOMPACT}

Theorem 1.21. *A topological space X is compact $\Leftrightarrow$ every ultrafilter on X has limit.*

[1] See also `http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/trf/nets/subnetdefs.pdf`

References Basic facts about convergence of filters and nets in topological spaces can be found in any standard topological textbook, e.g. [E]. (Although some authors include only one of these two approaches to convergence.) Nets are often useful in analysis, therefore the convergence of nets is quite frequently described in functional-analytic introductions, e.g. [M].

2 Relationships between limits of nets, filters and $\mathcal{F}$-limits

2.1 $\mathcal{F}$-limits and convergence of filters

We have already mentioned that convergence of a filter on a topological space X is a special case of $\mathcal{F}$-convergence. We next show that $\mathcal{F}$-convergence of a function $f\colon M \to X$ can be expressed using convergence of a suitable filter on X.

We start with the following easy observation.[2]

{LMFILTERBASE}

Lemma 2.1. *Let $f\colon X \to Y$ be a map, $\mathcal{F}$ be a filter on Y. Then*

$$\mathcal{G} = \{B \subseteq Y; f^{-1}(B) \in \mathcal{F}\}$$

is a filter on Y,

$$\mathcal{B} = \{f[A]; A \in \mathcal{F}\}$$

is a filterbase on Y and it generates $\mathcal{G}$.

Moreover, if $\mathcal{F}$ is an ultrafilter then $\mathcal{G}$ is an ultrafilter.

We will denote the above filterbase as $f[\mathcal{F}] := \{f[A]; A \in \mathcal{F}\}$.

Proof. $\mathcal{G}$ is a filter. $f^{-1}(\emptyset) = \emptyset \notin \mathcal{F}$, hence $\emptyset \notin \mathcal{G}$. Also $f^{-1}(Y) = X$, hence $Y \in \mathcal{G}$ and $\mathcal{G} \neq \emptyset$.

If $B \in \mathcal{G}$ and $B \subseteq C$, then we have $f^{-1}(B) \in \mathcal{F}$ and $f^{-1}(B) \subseteq f^{-1}(C)$. This implies $f^{-1}(C) \in \mathcal{F}$ and $C \in \mathcal{G}$.

If $B, C \in \mathcal{G}$ then $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C) \in \mathcal{F}$, and thus $B \cap C \in \mathcal{G}$.

$\mathcal{B}$ is a filterbase. For any $A, B \in \mathcal{F}$ we have $f[A] \cap f[B] \supseteq f[A \cap B]$. (And each $f[A]$ is non-empty.)

$\mathcal{B}$ generates $\mathcal{G}$. We have $f[f^{-1}(B)] \subseteq B$, and thus $\mathcal{G} \subseteq \mathcal{F}_{\mathcal{B}}$.

We also have $f^{-1}(f[A]) \supseteq A$, which means $\mathcal{B} \subseteq \mathcal{G}$.

If $\mathcal{F}$ is an ultrafilter, then $\mathcal{G}$ is an ultrafilter. $B \notin \mathcal{G} \Rightarrow f^{-1}(B) \notin \mathcal{F} \Rightarrow X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \in \mathcal{F} \Rightarrow Y \setminus B \in \mathcal{G}$. □

The $\mathcal{F}$-convergence of a function f is precisely the convergence of the filterbase $f[\mathcal{F}]$.

{THMFILTERBASE}

Theorem 2.2. *Let M be a set, $\mathcal{F}$ be a filter on M and $f\colon M \to X$ be a map from M to a topological space X. Let $x \in X$. Then f is $\mathcal{F}$-convergent to x if and only if the filterbase $f[\mathcal{F}]$ converges to x.*

Proof. Let $\mathcal{G}$ be the filter generated by $f[\mathcal{F}]$.

From Lemma 2.1 we have that $f^{-1}(U) \in \mathcal{F}$ and $U \in \mathcal{G}$ are equivalent. The validity of these claims for each neighborhood U of x is equivalent to $\mathcal{F}\text{-}\lim f = x$ and to $\mathcal{G} \to x$, respectively. □

We will show that several results concerning $\mathcal{F}$-limits can be easily obtained using Theorem 2.2 and that a similar correspondence works for several notions related to $\mathcal{F}$-limit.

[2] It is easy to show that similar claims hold for preimages; i.e. from a filter on X we can obtain a filter on M using a map $f\colon M \to X$. We do not include the proof of this fact, since we will not need it.

2.2 Convergence of nets as $\mathcal{F}$-convergence

The convergence of nets can also be interpreted as a special case of $\mathcal{F}$-convergence.

Definition 2.3. Let $(D, \leq)$ be a directed set. Then the system of all sets of the form

$$D_a = \{d \in D; d \geq a\}$$

if a *filterbase*. This filterbase will be denoted by $\mathcal{B}(D)$, i.e.

$$\mathcal{B}(D) = \{D_a; a \in D\}.$$

The filter generated by the filterbase $\mathcal{B}(D)$ will be denoted $\mathcal{F}(D)$ and called *section filter.*

Directly from the definition of the convergence of nets we can see that:

Theorem 2.4. *Lex X be a topological space and $x\colon D \to X$ be a net on a directed set $(D, \leq)$. Then the net x converges to a limit l in X if and only if $\mathcal{F}(\mathrm{D})\text{-}\lim x = l$.*

Terminology Apart from the name *section filter* (e.g. [Bo, p.60] and [AB, p.35,Section 2.6]), some authors use also the term *residual filter* [CSC, p.409]. The filter $x[\mathcal{F}(D)]$ is sometimes called *eventuality filter* [S, p.158] or *tail filter* [BP, p.171].

Theorem 2.2 is mentioned e.g. in [L, Príklad 3.2.35].

3 Basic properties of $\mathcal{F}$-limits

We will include several useful properties of $\mathcal{F}$-convergence. Many of them have been already mentioned for the two special cases of $\mathcal{F}$-convergence – the convergence of nets and filters in topological spaces.

Let us first mention how the $\mathcal{F}$-convergence relates to the usual convergence of sequences.

{PROPFINER}

Proposition 3.1. *Let $f\colon M \to X$ be a map from a set M to a topological space X. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filters on M such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. If x is $\mathcal{F}_1$-limit of f, then it is also $\mathcal{F}_2$-limit of x.*

$$\mathcal{F}_1\text{-}\lim f = x \qquad \Rightarrow \qquad \mathcal{F}_2\text{-}\lim f = x.$$

Proof. If x is a $\mathcal{F}_1$-limit of f then, by definition,

$$f^{-1}(U) \in \mathcal{F}_1 \subseteq \mathcal{F}_2$$

for each neighborhood U of x. This shows that x is also $\mathcal{F}_2$-limit of f. □

If we apply Proposition 3.1 to Fréchet filter $\mathcal{F}_0 = \{A \subseteq \mathbb{N}; \mathbb{N} \setminus A \text{ is finite}\}$ on the set $\mathbb{N}$, we get

Corollary 3.2. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ and $x\colon \mathbb{N} \to X$ be a sequence in a topological space X. If the sequence x converges to l, then it also $\mathcal{F}$-converges to l.*

{PROPHAUS}

Proposition 3.3. *Let X be a topological space, $\mathcal{F}$ be a filter on M and $f\colon M \to X$ be a function. If X is Hausdorff then f has at most one $\mathcal{F}$-limit.* [3]

[3] This would not be true if we allowed $\emptyset \in \mathcal{F}$, see Remark 1.2.

Proof. Suppose that x, y are both $\mathcal{F}$-limits of f and that $x \neq y$. Then there exist neighborhoods $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$. Since $f^{-1}(U) \in \mathcal{F}$ and $f^{-1}(V) \in \mathcal{F}$, we get that

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset \in \mathcal{F},$$

a contradiction. $\square$

As a special case of this result we get one half of Theorem 1.16 and one half of Theorem 1.20. Theorem 1.20 also shows that this is in fact characterization of Hausdorff spaces. We could also prove Proposition 3.3 from Theorem 1.20 using Theorem 2.2.

{PROPCONTI}

Proposition 3.4. *Let M be a set, $\mathcal{F}$ be a filter on M, $f\colon M \to X$ be a map and $g\colon X \to Y$ be a continuous map between two topological spaces. If $x \in \mathcal{F}\text{-}\lim f$, then $g(x) \in \mathcal{F}\text{-}\lim g \circ f$.*

Proof. Let x be an $\mathcal{F}$-limit of f, i.e., $f^{-1}(U) \in \mathcal{F}$ for each $U \in \mathcal{N}(x)$. Now if V is a neighborhood of $g(x)$, then by continuity $g^{-1}(V)$ is a neighborhood of x, which implies

$$f^{-1}(g^{-1}(V)) = g \circ f^{-1}(V) \in \mathcal{F}.$$

Since this is true for each $V \in \mathcal{N}(x)$, the point $g(x)$ is $\mathcal{F}$-limit of $g \circ f$. $\square$

In fact, it can be shown that preservation of (all) $\mathcal{F}$-limits is equivalent to continuity – this is true already for special cases by Theorems 1.15 and 1.19. Again, from Proposition 3.4 we could obtain one implication of these theorems and Proposition 3.4 could be obtained from Theorem 1.19 and Theorem 2.2. (See also [HS, Theorem 3.54].)

A special case when $\mathcal{F}$ is an ultrafilter is of particular importance.

{PROPCOMPACT}

Proposition 3.5. *Let $\mathcal{F}$ be an ultrafilter on M, X be a compact space and $f\colon M \to X$ be a map. Then $\mathcal{F}\text{-}\lim f$ exists.* (*More precisely, there exists at least one $\mathcal{F}$-limit.*)

Proof. Suppose that no point $x \in X$ is an $\mathcal{F}$-limit of f. Hence for every x there is a neighborhood U_x such that $f^{-1}(U_x) \notin \mathcal{F}$. By compactness, there is a finite subcover of $\{U_x; x \in X\}$.

Let us denote the sets from this subcover by $U_1, \dots, U_n$. For each $i = 1, \dots, n$ we have $f^{-1}(U_i) \notin \mathcal{F}$. Since $\mathcal{F}$ is ultrafilter, this is equivalent to $f^{-1}(X \setminus U_i) \in \mathcal{F}$.

Now $\bigcap_{i=1}^{n}(X \setminus U_i) = \emptyset$, since $U_1, \dots, U_n$ is a cover and this implies $\bigcap_{i=1}^{n} f^{-1}(X \setminus U_i) = f^{-1}(\bigcap_{i=1}^{n} X \setminus U_i) = \emptyset$. Consequently $\emptyset \in \mathcal{F}$, a contradiction. $\square$

As a special case of this result we could obtain one implication of Theorem 1.21. Conversely, we can deduce Proposition 3.5 from Theorem 1.21:

Proof. By Lemma 2.1 the filter given by the filterbase $f[\mathcal{F}]$ is an ultrafilter on X. Since X is compact and $f[\mathcal{F}]$ is an ultrafilter, there is a limit x of $f[\mathcal{F}]$ in X by Theorem 1.21. Then $x = \mathcal{F}\text{-}\lim f$. $\square$

In fact it can be shown that the property from Proposition 3.5 characterizes compact spaces - see Theorem 1.21.

{CORBOUNDEDSEQ}

Corollary 3.6. *Every bounded real sequence has $\mathcal{F}$-limit for any ultrafilter $\mathcal{F}$ on $\mathbb{N}$.*

{REMBETAM}

Remark 3.7. Perhaps it is worth mentioning that the above result is closely related to the Stone-Čech compactification of the discrete space on the set M. It is known that the set of all ultrafilters on M can be topologized in a such way that the resulting space βM contains M as a discrete subspace and every map f from M to a compact Hausdorff space X can be uniquely extended to a map from βM to X. The $\mathcal{F}$-limit of the map $\mathcal{F}$ is precisely the value of this unique extension in the point $\mathcal{F}$. Details can be found e.g. in [HS].

References Proposition 3.4 can be found e.g. in [Di, Theorem 2.3.3], [HS, Theorem 3.49], [GS, Lemma 2.3]. (In [GS] the term $\mathcal{F}$-limit point is used instead of $\mathcal{F}$-limit.)

Proposition 3.5 can be found e.g. in [Di, Theorem 4.3.5], [HS, Theorem 3.48], [T, p.64, Claim 14.1], [F, 2A3Se(i)].

4 Cluster points

4.1 Filters

{DEFCLUSTERFILTER}

Definition 4.1. A point x is called a *cluster point of a filter* $\mathcal{F}$ if x belongs to closure of every member of $\mathcal{F}$.

Definition 4.2. We say that a filter $\mathcal{F}'$ is *finer* than a filter $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$.

{THMFILTCLUSFINER}

Theorem 4.3. *A point x is a cluster point of a filter $\mathcal{F}$ if and only if there exists a filter $\mathcal{F}'$ which is finer than $\mathcal{F}$ and x is a limit of $\mathcal{F}$.*

Theorem 4.4. *If $\mathcal{F}$ is an ultrafilter and x is a cluster point of $\mathcal{F}$, then x is a limit of $\mathcal{F}$.*

Theorem 4.5. *A topological space X is compact $\Leftrightarrow$ every filter (every filterbase) in X has a cluster point.*

4.2 Nets

{DEFCLUSTERNET}

Definition 4.6. A point x is called a *cluster point of a net* $S = (x_d)_{d\in D}$ if for every $d_0 \in D$ there exists a $d \geq d_0$ such that $x_d \in U$.

$$(\forall U \in \mathcal{N}(x))(\forall d_0 \in D)(\exists d \geq d_0)x_d \in U \tag{2}$$ {EQ2}

The condition (2) is called (x_σ) is *frequently in U* or *cofinally in U.*

{THMNETCLUSFINER}

Theorem 4.7. *A point x is a cluster point of a net $(x_d)_{d\in D}$ if and only if there exists a subnet $(y_e)_{e\in E}$ of this net such that x is a limit of (y_e).*

Theorem 4.8. *A topological space X is compact $\Leftrightarrow$ every net in X has a cluster point.*

4.3 $\mathcal{F}$-cluster points

{DEFFCLUSTER}

Definition 4.9. Let X be a topological space, $\mathcal{F}$ be a filter on a set M and $f\colon M \to X$ be a map. A point $x \in X$ is said to be $\mathcal{F}$*-cluster point* of f if for every neighborhood U of x and for every $F \in \mathcal{F}$ the intersection $f^{-1}(U) \cap F$ is non-empty.

$$(\forall U \in \mathcal{N}(x))(\forall F \in \mathcal{F})f^{-1}(U) \cap F \neq \emptyset.$$

It is easy to see that if $\mathcal{F}$ is generated by a filterbase $\mathcal{B}$, then requiring $f^{-1}(U) \cap F \neq \emptyset$ for each $F \in \mathcal{B}$ gives an equivalent condition.

The cluster points of a net $x\colon D \to X$ are precisely the $\mathcal{F}(D)$-cluster points of x, since (2) is equivalent to

$$(\forall U \in \mathcal{N}(x))(\exists B \in \mathcal{B}(D))x^{-1}(U) \cap B \neq \emptyset.$$

Our definition of $\mathcal{F}$-cluster point is slightly different from the definition given in [KŠW]. Let us first show that the two definitions are equivalent. (This notion is defined in [KŠW] dually, using ideals instead of filters, and for a special case $M = \mathbb{N}$.)

{LMCLUSTERDEFID}

Lemma 4.10. *Let X be a topological space, $\mathcal{F}$ be a filter on a set M and $f\colon M \to X$ be a map. A point x is a cluster point of f if and only if for each neighborhood U of x $M \setminus f^{-1}(U) \notin \mathcal{F}$.*

$$(\forall U \in \mathcal{N}(x))(M \setminus f^{-1}(U) \notin \mathcal{F}).$$

Proof. Instead of the equivalence of these two conditions we can show that their negations are equivalent.

$$(\exists U \in \mathcal{N}(x))(\exists F \in \mathcal{F})(f^{-1}(U) \cap F = \emptyset) \Leftrightarrow (\exists U \in \mathcal{N}(x))(M \setminus f^{-1}(U) \in \mathcal{F}).$$

$\boxed{\Rightarrow}$ If $f^{-1}(U) \cap F = \emptyset$ then $F \subseteq M \setminus f^{-1}(U)$. If we assume that $F \in \mathcal{F}$, we get that $M \setminus f^{-1}(U) \in \mathcal{F}$.

$\boxed{\Leftarrow}$ Put $F := M \setminus f^{-1}(U)$. □

The following results shows the relationship between cluster points of filters and $\mathcal{F}$-cluster points.

Proposition 4.11. *Let X be a topological space, $\mathcal{F}$ be a filter on a set M and $f\colon M \to X$ be a map. A point $x \in X$ is $\mathcal{F}$-cluster point of f if and only if it is a cluster point of $f[\mathcal{F}]$.*

Proof. A point x a cluster point of $f[\mathcal{F}]$ if and only if it is in the closure of each $f[F]$, where $F \in \mathcal{F}$. This is true if and only if every neighborhood of x intersects F.

So we want to show the equivalence of the condition

$$(\forall U \in \mathcal{N}(x))(\forall F \in \mathcal{F})(U \cap f[F] \neq \emptyset), \tag{3}$$ {EQCLUSF}

which is equivalent to x being a cluster point of $f[\mathcal{F}]$, and

$$(\forall U \in \mathcal{N}(x))(\forall F \in \mathcal{F})(f^{-1}(U) \cap F \neq \emptyset),$$

which is the definition of $\mathcal{F}$-cluster point of f.

$\boxed{\Rightarrow}$ Suppose that $U \cap f[F] \neq \emptyset$. This implies that there exists a point $p \in F$ and such that $f(p) \in U$. This point belongs to $f^{-1}(U) \cap F$, hence $f^{-1}(U) \cap F \neq \emptyset$.

$\boxed{\Leftarrow}$ Assume that $f^{-1}(U) \cap F \neq \emptyset$. Since $f^{-1}(U \cap f[F]) = f^{-1}(U) \cap f^{-1}(f[F]) \supseteq f^{-1}(U) \cap F \neq \emptyset$, we get that $U \cap f[F] \neq \emptyset$. □

{PROPFCLUSULT}

Proposition 4.12. *Let X be a topological space, $\mathcal{F}$ be a filter on a set M and $f\colon M \to X$ be a map. A point $x \in X$ is $\mathcal{F}$-cluster point of f if and only if there exists an ultrafilter $\mathcal{G}$ on M such that $\mathcal{G} \supseteq \mathcal{F}$ and $\mathcal{G}$-$\lim f = x$.*

Proof. $\boxed{\Rightarrow}$ It suffices to show that $\mathcal{A} = \{f^{-1}(U) \cap F; F \in \mathcal{F}, U \in \mathcal{N}(x)\}$ has a finite intersection property and then take any ultrafilter $\mathcal{G}$ containing this system of sets.

If $F, G \in \mathcal{F}$ and $U, V \in \mathcal{N}(x)$ then $(f^{-1}(U) \cap F) \cap (f^{-1}(V) \cap G) = (f^{-1}(U) \cap f^{-1}(V)) \cap (F \cap G) = f^{-1}(U \cap V) \cap (F \cap G)$. Since $U \cap V$ is a neighborhood of X and $F \cap G \in \mathcal{F}$, the intersection is non-empty (by the definition of $\mathcal{F}$-cluster point).

$\boxed{\Leftarrow}$ For every neighborhood U of x we have $f^{-1}(U) \in \mathcal{G}$. If $F \in \mathcal{F} \subseteq \mathcal{G}$, then $f^{-1}(U) \cap F \in \mathcal{G}$, which implies that $f^{-1}(U) \cap F \neq \emptyset$. □

Corollary 4.13. *If $\mathcal{F}$ is an ultrafilter on M and x is an $\mathcal{F}$-cluster point of $f\colon M \to X$, then $x = \mathcal{F}$-$\lim f$.*

{CORCLUSTERISFLIM}

Corollary 4.14. *If $x\colon \mathbb{N} \to X$ is a sequence in a topological space X and ξ is a cluster point of the sequence x, then there exists an ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $\mathcal{F}$-$\lim x = \xi$.*

Combining Proposition 4.12 and Proposition 3.5 we get:

{CORCLUSNONEMPTY}

Corollary 4.15. *Let X be a compact topological space, $\mathcal{F}$ be a filter on a set M and $f\colon M \to X$ be a map. Then the set of all $\mathcal{F}$-cluster points of F is non-empty.*

{PROPCLOSINTERSECTION

Proposition 4.16. *Let X be a topological space, $\mathcal{F}$ be a filter on a set M and $f\colon M \to X$ be a map. The set of all $\mathcal{F}$-cluster points is equal to*

$$\bigcap_{F\in\mathcal{F}} \overline{f[F]}.$$

Proof. $x \in \bigcap_{F\in\mathcal{F}} \overline{f[F]} \Leftrightarrow (\forall U \in \mathcal{N}(x))(\forall F \in \mathcal{F}) U \cap f[F] \neq \emptyset \Leftrightarrow (\forall U \in \mathcal{N}(x))(\forall F \in \mathcal{F}) f^{-1}(U) \cap F \neq \emptyset$ $\square$

{CORCLUSCLOSED}

Corollary 4.17. *Let X be a topological space, $\mathcal{F}$ be a filter on a set M and $f\colon M \to X$ be a map. The set of all $\mathcal{F}$-cluster points of f is a closed subset of X.*

It is easy to see that $\mathcal{F}$-limit of f is an $\mathcal{F}$-cluster point of f. The following result shows when the converse is true.

{THMUNIQCLUSLIMIT}

Theorem 4.18. *Let X be a compact space, M be a set, $\mathcal{F}$ be a filter on M and $f\colon M \to X$ be a map. If a is unique $\mathcal{F}$-cluster point of f, then $a = \mathcal{F}\text{-}\lim f$.*

Proof. Let U be a neighborhood of X.

For any $y \notin U$ there is a neighborhood V_y such that $M \setminus f^{-1}(V) \in \mathcal{F}$. (Since y is not an $\mathcal{F}$-cluster point, there exists $F_y \in \mathcal{F}$ and V_y such that $f^{-1}(V_y) \cap F_y = \emptyset$. This implies $M \setminus f^{-1}(V_y) \supseteq F_y$ and $M \setminus f^{-1}(V_y) \in \mathcal{F}$.)

The open sets $\{U, V_y; y \notin U\}$ form an open cover of X. By compactness there is a finite subcover $\{U, V_{y_1}, \dots, V_{y_n}\}$. Denote $V := \bigcup_{i=1}^n V_{y_n}$. We know that $M \setminus f^{-1}(V) \in \mathcal{F}$. We also have $U \supseteq X \setminus V$, otherwise $\{U, V_{y_1}, \dots, V_{y_n}\}$ would not be a cover.

Thus we get $f^{-1}(U) \supseteq f^{-1}(X \setminus V) = M \setminus f^{-1}(V)$, which implies $f^{-1}(U) \in \mathcal{F}$. $\square$

References Information on cluster points of filters and nets can be found in many standard topological texts, e.g. [E].

$\mathcal{F}$-cluster point is defined in this way in [Bo, p.70–71]. (It is called cluster point with respect to the filter $\mathcal{F}$.) Dixmier [Di, Section 2.6] calls them *adherence values* along a filter (along a filter base).

Corollary 4.14 can be found in [GS, Lemma 2.3].

Proposition 4.16 can is formulated for filter bases in [Di, Theorem 2.6.6].

Theorem 4.18 is given e.g. in [Di, Corollary 4.2.4] as a consequence of a more general result [Di, Theorem 4.2.3]: If U is an open set which contains all $\mathcal{F}$-cluster points, then there exists and $F \in \mathcal{F}$ such that $\overline{f[F]} \subseteq U$.

5 Limit superior and limit inferior

In this section, we will be working with $\mathbb{R}$ instead of arbitrary topological space X. It would be more precise to say that we work with the compact space $X = \mathbb{R} \cup \{\pm\infty\}$, since here we allow $\mathcal{F}$-limit, $\mathcal{F}$-limit superior and $\mathcal{F}$-limit inferior to attain the infinite values too. But it seems to be more customary to write $\mathbb{R}$; so I kept this practise. But we are always tacitly assuming that we are, in fact, working in $\mathbb{R} \cup \{\pm\infty\}$.

5.1 Filters

For all other notions both $\mathcal{F}$-version for $f\colon M \to X$ and special case for $f = id_X$ have been studied. For the case of limit superior and inferior I did not find a book where limit superior inferior would be introduced for this special case.

5.2 Nets

{LMNETNONINCR}

Lemma 5.1. *Let $(x_d)_{d\in D}$ be a net of real numbers, which is non-increasing, i.e. $d \leq e \Rightarrow x_d \geq x_e$. Then*

$$\lim_{d\in D} x_d = \inf_{d\in D} x_d.$$

Proof. We would assume that x_d is bounded from bellow, i.e. $L := \inf_{d\in D} x_d$ is a real number. (It is easy to modify the proof for the case that this infimum is equal to $-\infty$.)

For each $\varepsilon > 0$ there exists d_0 with $L \leq x_{d_0} < L + \varepsilon$ (by definition of infimum). Monotonicity implies $L \leq x_d < L + \varepsilon$ for each $d \geq d_0$. $\square$

{DEFLIMSUPNET}

Definition 5.2. If $(x_d)_{d\in D}$ is a net of real numbers defined on a directed set $(D, \leq)$, then we define

$$\limsup x_d = \lim_{d\in D} \sup_{e\geq d} x_e = \inf_{d\in D} \sup_{e\geq d} x_e.$$

The fact, that the above limit exists and is equal to infimum, follows from the fact that the net $(\sup_{e\geq d} x_e)_{d\in D}$ is non-increasing and Lemma 5.1.

5.3 $\mathcal{F}$-limit superior

{DEFFLIMSUP}

Definition 5.3. For a function $f\colon M \to \mathbb{R}$ and a filter $\mathcal{F}$ on M we define $\mathcal{F}$-limit superior of f as

$$\mathcal{F}\text{-limsup}\, f = \sup\{a \in \mathbb{R}; (\forall F \in \mathcal{F}) f^{-1}((a,\infty)) \cap F \neq \emptyset\},$$

where $\sup \emptyset = -\infty$ by definition.

Similarly as in the case of $\mathcal{F}$-cluster points, this definition is slightly different from the one given in [Dem1, KMŠS], but these two definitions are equivalent. (The proof is almost identical to the one given in Lemma 4.10.)

{THMMAXCLUS}

Theorem 5.4. *Let $f\colon M \to \mathbb{R}$ be a map and $\mathcal{F}$ be a filter on M. Then $\mathcal{F}$-limsup f is the largest $\mathcal{F}$-cluster point of f.*

Proof. Let us denote $S := \mathcal{F}\text{-limsup}\, f$.

Since we are working in the compact space $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we know that the set of all $\mathcal{F}$-cluster points is a non-empty closed subset of $\overline{\mathbb{R}}$ (Corollary 4.15 and Corollary 4.17) hence there exists the maximal $\mathcal{F}$-cluster point C. It only remains to show that $C = S$.

We will assume that $S \in \mathbb{R}$, the proof for the case $S = +\infty$ can be done similarly.

Let x be an $\mathcal{F}$-cluster point of f. If $a < x$ then (a, ∞) is a neighborhood of x and thus $f^{-1}((a,\infty)) \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. This implies $x \leq S$.

Since the above inequality is true for each cluster point x, we get that $C \leq S$.

Now it only remains to show that S is an $\mathcal{F}$-cluster point of f. Suppose we are given a basic neighborhood $U = (S - \varepsilon, S + \varepsilon)$.

Since $a = S + \frac{\varepsilon}{2} > S$, we get from Definition 5.3 that there exists $G \in \mathcal{F}$ such that $f^{-1}((a,\infty)) \cap G = \emptyset$. This implies $f^{-1}((-\infty, S+\varepsilon)) \supseteq f^{-1}((-\infty, a\rangle) \supseteq G$, and thus $f^{-1}((-\infty, S+\varepsilon)) \in \mathcal{F}$.

Now let $F \in \mathcal{F}$. Then the intersection $F \cap f^{-1}((-\infty, S+\varepsilon))$ belongs to $\mathcal{F}$, too. Using Definition 5.3 again we get that

$$[F \cap f^{-1}((-\infty, S+\varepsilon))] \cap f^{-1}((S-\varepsilon, \infty)) = F \cap f^{-1}((S-\varepsilon, S+\varepsilon)) \neq \emptyset.$$

Since this is true for any $F \in \mathcal{F}$, we see that S is an $\mathcal{F}$-cluster point. $\square$

{CORSUPGLIM}

Corollary 5.5. *Let $f\colon M \to \mathbb{R}$ be a map and $\mathcal{F}$ be a filter on M. Then*

$$\mathcal{F}\text{-limsup}\, f = \sup\{\mathcal{G}\text{-lim}\, f; \mathcal{G} \supset \mathcal{F}, \mathcal{G} \textit{ is an ultrafilter}\}.$$

From Proposition 4.16 and Theorem 5.4 we can get another characterization of the $\mathcal{F}$-limit superior:

{PROPFLIMSUPINTERSECT

Proposition 5.6. *Let $f\colon M \to \mathbb{R}$ be a map and $\mathcal{F}$ be a filter on M. Then*

$$\mathcal{F}\text{-limsup}\, f = \sup \bigcap_{F \in \mathcal{F}} \overline{f[F]}.$$

{PROPLIMSUPLIMBASE}

Proposition 5.7. *Let $f\colon M \to \mathbb{R}$ be a map and $\mathcal{F}$ be a filter on M. Then*

$$\mathcal{F}\text{-limsup}\, f = \lim_{F \in \mathcal{F}} \sup f[F] = \inf_{F \in \mathcal{F}} \sup f[F],$$

i.e. $\mathcal{F}$-limsup f is limit of the net $(\sup f[F])_{F \in \mathcal{F}}$ on the directed set $(\mathcal{F}, \subseteq)$.

Proof. Let us denote $f_F := \sup f[F] = \sup \overline{f[F]}$. Since the net $(f_F)_{F \in \mathcal{F}}$ is non-increasing, the limit of this net exists and $\lim_{F \in \mathcal{F}} f_F = \inf_{F \in \mathcal{F}} f_F$.

Let us denote $L := \lim_{F \in \mathcal{F}} f_F$ and $S := \mathcal{F}$-limsup f.

$\boxed{L \leq S}$ If U is any open neighborhood of L then there exists $F_0 \in \mathcal{F}$ such that for each $F \in \mathcal{F}$, $F \subseteq F_0$ we have $\sup f[F] \in U$. This implies that

$$(\exists F_0 \in \mathcal{F}) G \in \mathcal{F}, G \subseteq F_0 \Rightarrow \overline{f[G]} \cap U \neq \emptyset.$$

Since U is open, we get that

$$(\exists F_0 \in \mathcal{F}) G \in \mathcal{F}, G \subseteq F_0 \Rightarrow f[G] \cap U \neq \emptyset.$$

Now for any $F \in \mathcal{F}$ we have $G := F \cap F_0 \subseteq F_0$, which implies $f[F] \cap U \neq \emptyset$. Hence there exists a point $x \in F$ such that $f(x) \in U$, which implies

$$F \cap f^{-1}(U) \neq \emptyset.$$

Thus L is an $\mathcal{F}$-cluster point of f and thus $L \leq S = \mathcal{F}$-limsup f by Theorem 5.4.

$\boxed{S \leq L}$ If $a < S$ then $(\forall F \in \mathcal{F}) f^{-1}((a, \infty)) \cap F \neq \emptyset$. This implies that

$$(\forall F \in \mathcal{F}) \sup f[F] > a$$

and thus $L = \lim_{F \in \mathcal{F}} \sup f[F] \geq a$.

Since the above inequality is true for each $a < S$, we get $L \geq S$. $\square$

It is easy to see that in Proposition 5.7 we can replace $F \in \mathcal{F}$ with $F \in \mathcal{B}$, if the filter $\mathcal{F}$ is generated by the filter base $\mathcal{B}$. If we now compare this proposition with Definition 5.2, we see that limit superior of a net is a special case of $\mathcal{F}$-limit superior.

Several inequalities related to $\mathcal{F}$-limit superior and inferior are similar to the properties of usual limit superior and inferior. It can be seen directly from the definition that $\mathcal{F}$-liminf $f \leq$ $\mathcal{F}$-limsup f. It is also clear that $\mathcal{F}$-limsup $f \leq \mathcal{F}$-limsup g for functions such that $f \leq g$.

{PROPFLIMINEQ}

Proposition 5.8. *Let $f, g\colon M \to \mathbb{R}$ be maps, $\mathcal{F}$ be a filter on M. Then*

$$\mathcal{F}\text{-liminf}\, f + \mathcal{F}\text{-liminf}\, g \le \mathcal{F}\text{-liminf}(f+g) \le \mathcal{F}\text{-limsup}(f+g) \le \mathcal{F}\text{-limsup}\, f + \mathcal{F}\text{-limsup}\, g.$$

If one of the functions has an $\mathcal{F}$-limit, than the above inequalities are equalities.

Proof. Let $A = \mathcal{F}\text{-limsup}(f+g)$. This implies that A is an $\mathcal{F}$-cluster point of $f+g$ and thus there is an ultrafilter $\mathcal{G} \supseteq \mathcal{F}$ such that $A = \mathcal{G}\text{-lim}(f+g)$.

Thus we get
$$A = \mathcal{G}\text{-lim}\, f + \mathcal{G}\text{-lim}\, g \le \mathcal{F}\text{-limsup}\, f + \mathcal{F}\text{-limsup}\, g,$$
since $\mathcal{G}\text{-lim}\, f$ is an $\mathcal{F}$-cluster point of f and hence it is less than or equal to $\mathcal{F}\text{-limsup}\, f$; similarly for g. □

The following result follows from Proposition 5.8, Theorem 5.4 and Theorem 4.18.

Proposition 5.9. *If $\mathcal{F}\text{-liminf}\, f = \mathcal{F}\text{-limsup}\, f = a$, then $\mathcal{F}\text{-lim}\, f = a$.*

{PROPFLIMSUPFINER}

Proposition 5.10. *Let $f\colon M \to \mathbb{R}$ be a map, $\mathcal{F}$, $\mathcal{G}$ be filters on M. If $\mathcal{G}$ is finer than $\mathcal{F}$, i.e. $\mathcal{F} \subseteq \mathcal{G}$, then*

$$\mathcal{F}\text{-liminf}\, f \le \mathcal{G}\text{-liminf}\, f \le \mathcal{G}\text{-limsup}\, f \le \mathcal{F}\text{-limsup}\, f.$$

Proof. TODO □

References Definition and some basic facts about lim sup and lim inf of nets can be found e.g. in [AB, p.32], [Be, p.2], [M, p.217], [S, 7.43–7.47]. (In fact, [S] deals with a more general case, where the values of the net are in a complete lattice.)

TODO filter

Various definitions of $\mathcal{F}\text{-limsup}\, f$ (a.k.a. limit superior with respect to filter/filter base, upper limit with respect to filter/filter base) can be found in the literature.

The definition given in [Bo, p.353, Definition 6] is the equivalent condition from Proposition 5.7. Proposition 5.6 is used as definition in [Ch, p.132, Definition 4.1]. It is defined in [Di, Section 7.3] as supremum of the set of all $\mathcal{F}$-cluster point, i.e. using our Theorem 5.4.

The $\mathcal{F}$-limit superior was studied e.g. in [Dem1], [KMŠS], [LD], [Š, Podkapitola 5.2.1].

TODO Theorem 5.4 is shown in [KMŠS, Theorem 5.3], [Š, Tvrdenie 5.2.4].

Proposition 5.10: [Bo, p.354, Corollary 2].

Some parts of Proposition 5.8 are shown e.g. in [Di, Theorems 7.3.6, 7.3.7]; [F, 2A3Sg].

6 Ultrafilters and ultranets

TODO We have already mentioned in Proposition 3.5...

TODO ultranets = universal nets

7 Cauchyness

Before discussing Cauchy filters and Cauchy nets, let us recall some basic facts about uniform spaces. (The reader can work with metric spaces instead of uniform spaces, if he prefers so. The changes in the proofs should be obvious.)

Definition 7.1. Let X be a set and Φ be a system of subsets of $X \times X$. The pair (X, Φ) is called *uniform space* if:

(i) $U \in \Phi \Rightarrow \Delta = \{(x, x); x \in X\} \subseteq U$;

(ii) $U \in \Phi \wedge V \supseteq U \Rightarrow V \in \Phi$;

(iii) $U, V \in \Phi \Rightarrow U \cap V \in \Phi$;

(iv) $U \in \Phi \Rightarrow (\exists V \in \Phi) V \circ V = \{(x, z); (\exists y \in X)(x, y), (y, z) \in V\} \subseteq U$;

(v) $U \in \Phi \Rightarrow U^{-1} \in \Phi$.

Elements of Φ are called *entourages* or *surroundings.*

A *fundamental system of entourages* is such a system $\mathcal{B} \subseteq \Phi$ that every entourage $U \in \Phi$ contains an element of $\mathcal{B}$.

For every $U \in \Phi$ and $x \in X$ we denote

$$U[x] := \{y \in X; (x, y) \in U\}.$$

Fundamental system of entourages has a similar role as base in topological spaces. It is sufficient to determine the uniformity.

Note that every uniformity on X is a filter on $X \times X$.

The system $U[x]$, $U \in \Phi$ is a local base for the topology of the uniform space (U, Φ). A topological space is uniformizable if and only if it is completely regular.

Standard examples of uniform spaces are metric spaces, where a fundamental system of entourages consists of the sets of the form $\{(x, y); d(x, y) < \varepsilon\}$ for $\varepsilon > 0$, and topological groups, with the fundamental system of entourages given by $\{(x, y); xy^{-1} \in U\}$, U being a neighborhood of 0.

Several equivalent definitions of uniform spaces can be found in the literature – using families of pseudometrics, using uniform covers.

7.1 Cauchy filters

Definition 7.2. Let X be a uniform space and $\mathcal{F}$ be a filter on X. The filter $\mathcal{F}$ is called *Cauchy,* if for every entourage U there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$ (i.e., F is U-small).

Uniform space is called *complete* if every uniform filter converges.

Uniform space X is called *totally bounded* if for every entourage U there exists a finite set $\{x_1, \ldots, x_n\}$ such that $X = \bigcup_{i=1}^{n} U[x_i]$.

A uniform space is compact if and only if it is complete and totally bounded. TODO Reference [W, Theorem 39.9]

7.2 Cauchy nets

Definition 7.3. Let X be a uniform space and $(x_d)_{d \in D}$ be a net in X. The net $(x_d)_{d \in D}$ is said to be a *Cauchy net,* if for each entourage U there exists $d_0 \in D$ such that

$$d, e \geq d_0 \Rightarrow (x_d, x_e) \in U.$$

TODO X is totally bounded if and only if each net has Cauchy subnet. [W, Lemma 39.8]

7.3 $\mathcal{F}$-Cauchyness

Lemma 7.4. *Let (X, Φ) be a uniform space, $\mathcal{F}$ be an filter of M. Let $f\colon M \to X$ be a function. For $U \in \Phi$ and $m \in M$ we denote $E_m(U) = \{n \in M; (f(n), f(m)) \in U\}$ The following conditions are equivalent:* {LMFCAUCHY}

(i) $(\forall U \in \Phi)(\exists F \in \mathcal{F})\{(f(x), f(y)); x, y \in F\} \subseteq U$*;* {itemFILT}

(ii) $(\forall U \in \Phi)\{m \in M; E_m(U) \in \mathcal{F}\} \in \mathcal{F}$*;* {itemIK}

(iii) $(\forall U \in \Phi)(\exists m \in M)\{n \in M : (f(n), f(m)) \in U\} \in \mathcal{F}$*;* {itemCAUCH}

Proof. (i) $\Rightarrow$ (ii): Choose any $m \in F$. From (i) we get $E_m(U) \supseteq F$ and $E_m(U) \in \mathcal{F}$. Thus $\{m \in M; E_m(U) \in \mathcal{F}\} \supseteq F$, which implies $\{m \in M; E_m(U) \in \mathcal{F}\} \in \mathcal{F}$.

(ii) $\Rightarrow$ (iii): Obviously, (iii) is equivalent to $\{m \in M; E_m(U) \in \mathcal{F}\} \neq \emptyset$.

(iii) $\Rightarrow$ (i) Let $V \in \Phi$ be entourage such that $V \circ V \subseteq U$. By (iii) there exists $m \in M$ such that $F := \{n \in M; (f(n), f(m)) \in V \cap V^{-1}\}$ belongs to $\mathcal{F}$. Now for any $x, y \in F$ we get that both $(f(x), f(m)) \in V$ and $(f(y), f(m)) \in V$, since $(f(m), f(y)) \in V^{-1}$. This implies $(f(x), f(y)) \in V \circ V \subseteq U$. $\square$

Definition 7.5. Let (X, Φ) be a uniform space and $\mathcal{F}$ be an ultrafilter on M. We will say that a function $f\colon M \to X$ is *$\mathcal{F}$-Cauchy* if it fulfills any of the equivalent conditions from Lemma 7.4.

If we compare condition (i) with the definition of Cauchy net, we see that a net $(x_d)_{d \in D}$ is Cauchy if and only if the function $x\colon D \to X$ is $\mathcal{F}(D)$-Cauchy.

From condition (i) we also get:

Proposition 7.6. *Let (X, Φ) be a uniform space, $\mathcal{F}$ be an filter of M. Let $f\colon M \to X$ be a function. The function f is $\mathcal{F}$-Cauchy if and only if the filter generated by $f[\mathcal{F}]$ is Cauchy.*

In particular, for the special case $id_X\colon X \to X$ (and a filter $\mathcal{F}$ on X) we obtain precisely the notion of Cauchy filter.

Proposition 7.7. *Let (X, Φ) be a uniform space and $\mathcal{F}$ be a filter on M. If a function $f\colon M \to X$ is $\mathcal{F}$-convergent, then it is $\mathcal{F}$-Cauchy.*

Proof. Let $U \in \Phi$, V be an entourage such that $V \circ V \subseteq U$ and $W := V \cap V^{-1}$.

Let $\mathcal{F}\text{-}\lim f = x$.

Put $F := f^{-1}(W[x])$. The set $F \in \mathcal{F}$ fulfills (i).[4] $\square$

Corollary 7.8. *Let (X, Φ) be a uniform space. If $\mathcal{F}$ is an ultrafilter on M, then every function $f\colon M \to X$ is $\mathcal{F}$-Cauchy.*

In particular, every ultrafilter on a compact uniform space is a Cauchy filter.

TODO Complete $\Leftrightarrow$ Every $\mathcal{F}$-Cauchy function is convergent.

References TODO Uniform space: many topological texts TODO [J]; some texts in functional analysis TODO

Lemma 7.4 - see [Dem2, Proposition 4]

For the filter $\mathcal{F}_d = \{A \subseteq \mathbb{N}; d(A) = 1\}$ of sets having asymptotic density 1, the condition (iii) is called statistically Cauchy and (i) is called statistically quasi-Cauchy in [DMK].

[4] $y \in F \Rightarrow (x, f(y)) \in W^{-1} \subseteq V^{-1} \Rightarrow (f(y), x) \in V$
$z \in F \Rightarrow (x, f(z) \in W \subseteq V)$
For any $y, z \in F$, we have $(f(y), x), (x, f(z)) \in V$, which implies $(f(y), f(z)) \in V \circ V \subseteq U$.

8 The notions which are not expressible using $f[\mathcal{F}]$

TODO In this part with countable sets (sequences) only

TODO Intro

{EXALETAVAJ}

Example 8.1. We will use in several examples the following two filters and functions.

Let X be a Hausdorff space. Let $x\colon \mathbb{N} \to X$ be any injective sequence, which converges to $\xi \in X$ in the usual sense.

Let $\mathcal{F}_2$ be the filter on $\mathbb{N}\times\mathbb{N}$ generated by the filterbase

$$\mathcal{B}_2 = \{p_1^{-1}(A); A \in \mathcal{F}_0\},$$

where $p_1\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, $p_1(a,b) = a$.

In the other words, A belongs to $\mathcal{F}_2$ if and only if there exists $n_0 \in \mathbb{N}$ such that $(n,k) \in A$ for each $n \geq n_0$.

TODO bijection $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$; Relation to $\mathcal{I}_m$ from [KŠW]

It is easy to see that the filterbases $x[\mathcal{F}_0]$ and $x \circ p_1[\mathcal{F}_2]$ generate the same filter on the space X.

8.1 $\mathcal{F}^*$-convergence

Definition 8.2. Let $f\colon M \to X$ be a map from a countable set M to a topological space X. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. We say that f is *$\mathcal{F}^*$-convergent* to $\xi \in X$ if for there exists a subset $S \in \mathcal{F}$ such that $f|_S$ converges in the usual sense to ξ; i.e., for every neighborhood U of ξ the set $f^{-1}(U) \cap S$ is cofinite in S.

It is known that $\mathcal{F}$-convergence and $\mathcal{F}^*$-convergence of sequences is equivalent if and only if $\mathcal{F}$ is a P-filter (the dual ideal is P-ideal). The same is true for sequences in metric space or, more generally, first countable spaces. (The name P-ideal seems to be more common than P-filter, but from the two approaches we have chosen to use filters rather than ideals in this text. Ultrafilters, which are P-filters, are called p-points.)

The sequence $x\colon \mathbb{N} \to X$ is convergent in usual sense, hence it is also $\mathcal{F}^*$-convergent.

However, $x \circ p_1$ is not $\mathcal{F}^*$-convergent to ξ. (Let $M \in \mathcal{F}_2$. Then there exists n such that $\{n\} \times \mathbb{N} \in M$ and a neighborhood U of ξ such that $x_n \notin U$. Since the set $(x \circ p_1)^{-1}(U)$ is infinite. $x \circ p_1|_M$ does not converge to ξ.)

8.2 $\mathcal{F}^*$-Cauchy sequences

Definition 8.3. Let $f\colon M \to X$ be a map from a countable set M to a uniform space X. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. We say that f is *$\mathcal{F}^*$-Cauchy* if there exists a set $S \in \mathcal{F}$ such that $f|_S$ is Cauchy in the usual sense, i.e., for each entourage U there exists a finite set $F \subseteq S$ such that $(f(m), f(n)) \in U$ for each $m, n \notin F$.

If the uniform space X is complete then, obviously, a sequence is $\mathcal{F}^*$-Cauchy if and only if it is $\mathcal{F}^*$-convergent. Hence Example 8.1 with $X = \mathbb{R}$ shows that $\mathcal{F}^*$-Cauchyness cannot be expressed using $f[\mathcal{F}]$.

8.3 $\mathcal{F}$-Limit points

Definition 8.4. Let $f\colon M \to X$ be a map from a countable set M to a topological space X. A point $\xi \in X$ is said to be *$\mathcal{F}$-limit point of f* if there exists $S \in \mathcal{F}$ such that $f|_S$ converges in the usual sense to ξ; i.e., for every neighborhood U of ξ the set $f^{-1}(U) \cap S$ is cofinite in S.

Again, for the functions x and $x \circ p_1$ from Example 8.1 we get that ξ is an $\mathcal{F}$-limit point of x but not an $\mathcal{F}$-limit point of $x \circ p_1$. This can be justified in a similar way as in the case of $\mathcal{F}^*$-convergence.

References TODO P-filters [KŠW]

$\mathcal{F}$-cluster points and $\mathcal{F}$-limit points were introduced in [KŠW] as a generalization of statistical cluster points.

TODO Example 8.1 [L, Príklad 3.2.38, 3.2.39]

$\mathcal{F}^*$-Cauchy sequences were studied in [NPG].

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