Notes on lower semicontinuous submeasures

References: [F], [K, Section 3.3].

Definitions and basic properties

Definition 1. A *submeasure* on $\mathbb{N}$ is a function $\varphi\colon \mathcal{P}(\mathbb{N}) \to \langle 0, +\infty\rangle$ such that

$$\varphi(\emptyset) = 0$$
$$\varphi(A) \le \varphi(A \cup B) \le \varphi(A) + \varphi(B)$$

Clearly, we can work with any countable set instead of $\mathbb{N}$.

Notice that the second property also means that $A \subseteq B$ implies $\varphi(A) \le \varphi(B)$. In the other words, a submeasure is a function $\varphi\colon \mathcal{P}(\mathbb{N}) \to \langle 0, +\infty\rangle$ which is monotone and subadditive.

Definition 2. A submeasure φ is called *lower semicontinuous* if

$$\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [1, n])$$

for every $A \subseteq \mathbb{N}$.

Authors using ordinal notation often write use $n = [0, n-1]$, which might lead to a more compact notation, especially when working with ω instead of $\mathbb{N}$.[1] I will mostly stick to the interval notation here.

If φ is a lsc submeasure, then we can define

$$\|A\|_\varphi = \limsup_{n\to\infty} \varphi(A \setminus [1, n]) = \lim_{n\to\infty} \varphi(A \setminus [1, n]).$$

The function $\|\cdot\|_\varphi$ is a submeasure (not necessarily lsc).

Ideals from submeasures

It can be easily seen that for any submeasure φ the sets

$$\operatorname{Fin}(\varphi) = \{A \subseteq \mathbb{N}; \varphi(A) < \infty\}$$
$$\operatorname{Nul}(\varphi) = \{A \subseteq \mathbb{N}; \varphi(A) = 0\}$$

are ideals.

Since $\|\cdot\|_\varphi$ is also a submeasure, we get that also

$$\operatorname{Exh}(\varphi) = \{A \subseteq \mathbb{N}; \|A\|_\varphi = 0\}$$

is an ideal.

We will need $\varphi(\{n\}) < +\infty$ for $n \in \mathbb{N}$ if we want $\operatorname{Fin}(\varphi)$ to be admissible. Similarly, $\operatorname{Nul}(\varphi)$ will be admissible iff $\varphi(\{n\}) = 0$ for each n. We may notice that $\operatorname{Exh}(\varphi)$ is always admissible, since $\|\{n\}\|_\varphi = 0$.

[1] For example, we could write $\varphi(A) = \lim\limits_{n\to\infty} \varphi(A \cap n)$ in the definition of lower semicontinuity. Or we could use $\|A\|_\varphi = \lim\limits_{n\to\infty} \varphi(A \setminus n)$.

Summability ideals

Erdös-Ulam ideals

Analytic P-ideals

TODO [F, Theorem 1.2.5]

References

[F] Ilijas Farah. Analytic quotients. *Mem. Amer. Math. Soc.*, 148(702), 2000.

[K] Vladimir Kanovei. *Borel Equialence Relations. Structure and Classification.* AMS, Providence, 2008.