Notes on lower semicontinuous submeasures

References: [F], [K, Section 3.3].

Topology on $\mathcal{P}(\mathbb{N})$ as $\{0,1\}^{\mathbb{N}}$

We will often identify $\mathcal{P}(\mathbb{N})$ with $\{0,1\}^{\mathbb{N}}$ using the bijection $A \mapsto \chi_A$. The product space $\{0,1\}^{\mathbb{N}}$ is also known as *Cantor cube*. (Here $\{0,1\}$ is endowed with the discrete topology, so $\{0,1\}^{\mathbb{N}}$ is product of countably many two-point discrete spaces.)

In this way we get a metrizable topology on $\mathcal{P}(\mathbb{N})$. (In fact, it is compact and completely metrizable.) Which means that we can use some topological notions for subsets of $\mathcal{P}(\mathbb{N})$ (in particular, for ideals). For example, it makes sense to ask whether an ideal is F_σ, Borel, etc.[1]

A local basis at a set A consists of sets

$$\{B \subseteq \mathbb{N}; B \cap F = A \cap F\}$$

for $F \subseteq \mathbb{N}$ finite.

Product topology is precisely the topology of pointwise convergence. This means that a net A_λ converges to A if and only if

$$\chi_{A_\lambda}(x) \to \chi_A(x)$$

for each $x \in \mathbb{N}$.

Basic definitions

Submeasure:
- $\varphi(\emptyset) = 0$;
- $\varphi(A) \le \varphi(A \cup B) \le \varphi(A) + \varphi(B)$;

A submeasure is *lower semicontinuous* if

$$\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [1,n]).$$

Lower-semicontinuous on $\{0,1\}^{\mathbb{N}}$

Proposition 1. *Let φ be a submeasure on $\mathbb{N}$. Then φ is lower semicontinuous (as a submeasure) if and only if the corresponding function $\{0,1\}^{\mathbb{N}} \to \langle 0,\infty\rangle$ is lower semicontinuous w.r.t. the product topology.*

Recall that a function $f\colon X \to \mathbb{R}$ is lower semicontinuous iff

$$f^{-1}((a,\infty)) = \{x \in X; f(x) > a\}$$

is open for every $a \in \mathbb{R}$.

[1] An this seems to be standard approach – if you encounter some text dealing with ideals that are F_σ, Borel, analytic; if the topology is not specified, the authors probably mean this topology.

This is equivalent to validity of

$$f(p) \le \liminf_{x\to p} f(x)$$

for every $p\in X$. If X is a metric space, it suffices to require

$$f(p) \le \liminf_{n\to\infty} f(x_n)$$

for every sequence $x_n \to p$.

Proof. In the proof we are identifying $\mathcal{P}(\mathbb{N})$ with $\{0,1\}^{\mathbb{N}}$ anyway; we will also use φ both for the submeasure (=function $\mathcal{P}(\mathbb{N}) \to \langle 0,\infty\rangle$) and the corresponding function $\varphi\colon \{0,1\}^{\mathbb{N}} \to \langle 0,\infty\rangle$.

$\boxed{\Rightarrow}$ Let a be a real number and $\varphi(A)>a$.

From the semicontinuity of the submeasure φ we get that there exists n_0 such that

$$\varphi(A\cap[1,n])>a$$

for each $n\ge n_0$. Therefore the set of all sets $B\subseteq\mathbb{N}$ such that

$$B\cap[1,n_0]=A\cap[1,n_0]$$

is a neighborhood $\mathcal{U}$ of A (in the product topology) such that $\varphi(B)>a$ for each $B\in\mathcal{U}$.

$\boxed{\Leftarrow}$ The sequence $A\cap[1,n]$ converges to A in the product topology. So we have

$$\varphi(A) \le \liminf_{n\to\infty} \varphi(A\cap[1,n])$$

from lower semicontinuity. But since $A\cap[1,n]\subseteq A$, we also get

$$\limsup_{n\to\infty} \varphi(A\cap[1,n]) \le \varphi(A)$$

from monotonicity. □

It is useful to notice that a lsc submeasure is in fact σ-subadditive:

Proposition 2. *Let φ be a lsc submeasure. For any sets $A_i\subseteq\mathbb{N}$, $i=1,2,\dots,$ we have*

$$\varphi\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_{i=1}^\infty \varphi(A_i).$$

Proof. Let

$$B_n=\bigcup_{i=1}^n A_i$$
$$A=\bigcup_{i=1}^\infty A_i$$

Notice that in the product topology we have $\lim\limits_{n\to\infty} B_n = A$. (It is an increasing sequence of sets.)

From lower semicontinuity we get

$$\varphi(A) \le \liminf_{n\to\infty} \varphi(B).$$

At the same time we have

$$\varphi(B) \leq \sum_{i=1}^{n} \varphi(A_i)$$

which together gives

$$\varphi(A) \leq \liminf_{n\to\infty} \varphi(A_i) = \sum_{i=1}^{\infty} \varphi(A_i).$$

□

Proof. Let

$$A = \bigcup_{i=1}^{\infty} A_i.$$

Fix some $n \in \mathbb{N}$. Then we clearly have

$$A \cap [1,n] = \bigcup_{i=1}^{\infty} (A_i \cap [1,n]).$$

However, since $A \cap [1,n]$ is finite, there is a finite set F such that

$$A \cap [1,n] = \bigcup_{i \in F} (A_i \cap [1,n]).$$

Using *finite* subadditivity we get

$$\varphi(A \cap [1,n]) \leq \sum_{i \in F} \varphi(A_i) \leq \sum_{i=1}^{\infty} \varphi(A_i).$$

Since φ is lower semicontinuous, we also get

$$\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [1,n]) \leq \sum_{i=1}^{\infty} \varphi(A_i).$$

□

If we have Proposition 1, we can use known properties of lower semi-continuous functions on metric spaces.

Corollary 1. *Supremum of a set of submeasures is again a submeasure.*
Supremum of a set of lsc submeasures is again a lsc submeasure.

Proof. Let φ_i be a submeasure for each $i \in I$ and let

$$\varphi(A) = \sup_{i \in I} \varphi_i(A_i).$$

Since $\varphi_i(\emptyset) = 0$ for each $i \in I$, we get

$$\varphi(\emptyset) = \sup_{i \in I} \varphi_i(\emptyset) = 0.$$

Similarly, if $\varphi_i(A) \leq \varphi_i(A \cup B) \leq \varphi_i(A) + \varphi_i(B)$ for each $i \in I$, then

$$\sup_{i \in I} \varphi_i(A) \leq \sup_{i \in I} \varphi_i(A \cup B) \leq \sup_{i \in I}(\varphi_i(A) + \varphi_i(B)) \leq \sup_{i \in I} \varphi_i(A) + \sup_{i \in I} \varphi_i(B)$$

which gives monotonicity and subadditivity of φ.

If each φ_i is a lsc submeasure, then so is $\varphi = \sup_{i \in I} \varphi_i$. □

Operations with submeasures

We have seen in Corollary 1 that submeasures (and lsc submeasures) are closed under arbitrary suprema. It is natural to ask what other operations can produce submeasures (lsc submeasures).

It is easy to check that if $\varphi_{1,2}$ are submeasures, then so is $\varphi_1+\varphi_2$. Also if $c\geq 0$, then $c\varphi_1$ is submeasure. The same is true for lsc submeasures. (Since lower semicontinuous functions are close under finite sums, finite minima and non-negative scalar multiples.)

Lemma 1. *Let (φ_n) be a sequence of submeasures such that for every set $A\subseteq\mathbb{N}$ the limit $\lim\limits_{n\to\infty}\varphi_n(A)$ exists. Then*

$$\varphi(A)=\lim_{n\to\infty}\varphi_n(A)$$

is also a submeasure.

Proof. Trivial. □

The function $\|\cdot\|_\varphi$ is a submeasure

If φ is a submeasure, then we can define

$$\|A\|_\varphi=\limsup_{n\to\infty}\varphi(A\setminus[1,n])=\lim_{n\to\infty}\varphi(A\setminus[1,n]).$$

Notice that the sequences $(\varphi(A\setminus[1,n]))$ is non-increasing, which implies that the above limit exists. (Hence it is equal to limit superior.)

Proposition 3. *If φ is a submeasure, then also the function $\|\cdot\|_\varphi$ defined above is a submeasure.*

Proof. It is relatively easy to see that $A\mapsto\varphi(A\setminus[1,n])$ is a submeasure for each n.

The limit of these submeasures $\|\cdot\|_\varphi$ is also a submeasure by Lemma 1. □

Examples

Examples of lsc submeasures

Example 1. For any $x\in\mathbb{N}$ we denote

$$\delta_x(A)=\begin{cases}1, & x\in A,\\ 0, & x\notin A.\end{cases}$$

In the other words, $\delta_x(A)=\chi_A(x)$.

It is easy to check that

$$\delta_x(A)\leq\delta_x(A\cup B)\leq\delta_x(A)+\delta_x(B)$$

for any $A,B\subseteq\mathbb{N}$.

Moreover, the functions $A\mapsto\delta_x(A)$ is continuous w.r.t. the product topology. (It is the projection onto x-th coordinate.)

It is also clear that

$$\delta_x(A)=\lim_{n\to\infty}\delta_x(A\cap[1,n]).$$

Both arguments mentioned above show that δ_x is a *lsc submeasure.*

We may notice that in this case we get $\|A\|_{\delta_x}=0$ for each A.

Example 2. Another simple example is *counting measure*

$$\varphi(A) = |A|.$$

Clearly, we have $\varphi(A) = \lim\limits_{n\to\infty} \varphi(A\cap[1,n])$, so this is a lsc submeasure. (In fact, it is additive, not only subadditive.)

We get that

$$\|A\|_\varphi = \begin{cases} 0 & \text{if } A \text{ is finite,} \\ +\infty & \text{if } A \text{ is infinite.} \end{cases}$$

So we can notice that in this case $\|\cdot\|_\varphi$ is not lower semicontinuous.

Counterexamples

- The function $\|\cdot\|_\varphi$ need not be lower semicontinuous: Example 2.
- A submeasure which is not lsc: $\|\cdot\|_\varphi$ from Example 2.
- A submeasure which is not countably subadditive: $\|\cdot\|_\varphi$ from Example 2.

Ideals

Basic properties of $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$

For any submeasure φ we can define

$$\begin{aligned} \mathrm{Fin}(\varphi) &= \{A \subseteq \mathbb{N}; \varphi(A) < \infty\} \\ \mathrm{Nul}(\varphi) &= \{A \subseteq \mathbb{N}; \varphi(A) = 0\} \\ \mathrm{Exh}(\varphi) &= \{A \subseteq \mathbb{N}; \|A\|_\varphi = 0\} \end{aligned}$$

It is easy to check that they are ideals.

Proposition 4. *Let φ be a lsc submeasure. Then*

- $\mathrm{Exh}(\varphi)$ *is* $F_{\sigma\delta}$*;*
- $\mathrm{Fin}(\varphi)$ *is* F_σ*.*

If, additionally, $\varphi(\{n\}) < +\infty$ for each n, then

$$\mathrm{Exh}(\varphi) \subseteq \mathrm{Fin}(\varphi).$$

Proof. For any fixed n the set $K_n = \{A \subseteq \mathbb{N}; \varphi(A) \le n\}$ is closed and thus

$$\mathrm{Fin}(\varphi) = \bigcup_{k=1}^{\infty} K_n$$

is a F_σ set.

For any $m, n \in \mathbb{N}$ the set $L_{m,n} = \{A \subseteq \mathbb{N}; \varphi(A \setminus [1,n]) \le 1/m\}$ is closed. We have

$$\mathrm{Exh}(\varphi) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} L_{m,n}.$$

Just notice that

$$\|A\|_\varphi = 0 \qquad \Leftrightarrow \qquad (\forall m)(\exists n)\varphi(A \setminus [1,n]) < \frac{1}{m}.$$

(Here we are also using the fact that $\varphi(A \setminus [1,n])$ is non-increasing.)

So we get that $\operatorname{Exh}(\varphi)$ is $F_{\sigma\delta}$.

It remains to show that $\operatorname{Exh}(\varphi) \subseteq \operatorname{Fin}(\varphi)$ (assuming $\varphi(\{n\}) < +\infty$, which immediately gives that $\varphi(F) < +\infty$ for F finite.)

If we have $\|A\|_\varphi = 0$, this means that $\varphi(A \setminus [1,n]) < +\infty$. If we also have that $\varphi([1,n])$ is finite, then

$$\varphi(A) \le \varphi(A \setminus [1,n]) + \varphi([1,n]) < +\infty$$

and $A \in \operatorname{Fin}(\varphi)$. □

The ideal $\operatorname{Exh}(\varphi)$ is a P-ideal

See also: [F, Lemma 1.2.2].

Proposition 5. *If φ is a lsc submeasure then* $\operatorname{Exh}(\varphi)$ *is a P-ideal.*

Proof. Suppose that we have a sequence of sets $A_1, A_2 \dots \in \operatorname{Exh}(\varphi)$. We want to show that there is $A \in \operatorname{Exh}(\varphi)$ such that $A_k \subseteq^* A$ for every k.

Since $\|A_k\|_\varphi = 0$, we can choose an n_k such that

$$\varphi(A_k \setminus [1,n_k]) \le \frac{1}{2^{k+1}}.$$

Then we put

$$A = \bigcup_{k=1}^{\infty} (A_k \setminus [1,n_k]).$$

It is clear that $A_k \subseteq^* A$. We want to show that also $\|A\|_\varphi = 0$.

Let us fix $n \in \mathbb{N}$. We know that

$$\bigcup_{k=1}^{n} A_k \in \operatorname{Exh}(\varphi)$$

since $\operatorname{Exh}(\varphi)$ is an ideal. This means that for large enough m we have

$$\varphi\left(\bigcup_{k=1}^{n} A_k \setminus [1,m]\right) \le \frac{1}{2^{n+1}}.$$

From countable subadditivity (Proposition 2) we also get

$$\varphi\left(\bigcup_{k=n+1}^{\infty} (A_k \setminus [1,n_k])\right) \le \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2^{n+1}}.$$

Together we get that

$$\varphi(A \setminus [1,m]) \le \varphi\left(\bigcup_{k=1}^{n} A_k \setminus [1,m]\right) + \varphi\left(\bigcup_{k=n+1}^{\infty} (A_k \setminus [1,n_k])\right) \le \frac{1}{2^n}.$$

Since this is true for arbitrary n, we get

$$\|A\|_\varphi = \lim_{m\to\infty} \varphi(A \setminus [1,m]) = 0.$$

□

Example 3. We may notice that $\mathrm{Fin}(\varphi)$ is not necessarily a P-ideal.

Let $\{A_i; i \in \mathbb{N}\}$ be a decomposition of $\mathbb{N}$ into countably many infinite sets. If is well known that

$$\mathcal{I} = \{A \subseteq \mathbb{N}; A \text{ intersects only finitely many } A_i\text{'s}\}$$

is not a P-ideal.

We have $\mathcal{I} = \mathrm{Fin}(\varphi)$ for

$$\varphi(A) = \frac{1}{\min\{i; A \cap A_i \neq \emptyset\}}.$$

(Using the convention $\min\emptyset = \infty$ and $1/\infty = 0$.)

This is indeed a submeasure. To show that $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ it suffices to observe that

$$\frac{1}{a} \leq \frac{1}{\min(a,b)} \leq \frac{1}{a} + \frac{1}{b}$$

(where $a = \min\{i; A \cap A_i \neq \emptyset\}$ and $b = \min\{i; B \cap A_i \neq \emptyset\}$).

And it is also lower semicontinuous, since the sequence given by

$$n \mapsto \min\{i; A \cap [1,n] \cap A_i \neq \emptyset\}$$

is eventually constant

Summable ideals

Let $f\colon \mathbb{N} \to \mathbb{R}^+$ be any function such that $\sum\limits_{i\in\mathbb{N}} f(i) = \infty$. Let us define

$$\mu_f(A) = \sum_{i\in A} f(i)$$

Since we can rewrite μ_f as

$$\mu_f = \sup\{\sum_{x\in F} f(i)\delta_x; F \subseteq \mathbb{N} \text{ is finite}\};$$

$$\mu_f = \sup\{\sum_{k=1}^{n} f(i)\delta_k; n \in \mathbb{N}\};$$

we see that μ_f is a lsc submeasure by Corollary 1.

Then we have an ideal

$$\mathcal{I}_f = \mathrm{Fin}(\mu_f) = \{A \subseteq \mathbb{N}; \sum_{i\in A} f(i) < +\infty\}.$$

It is an F_σ ideal. The following observation shows that it is a P-ideal.

Lemma 2. *For any $f\colon \mathbb{N} \to \mathbb{R}$ we have*

$$\mathrm{Fin}(\mu_f) = \mathrm{Exh}(\mu_f).$$

Proof. From Proposition 4 we already know that $\mathrm{Exh}(\mu_f) \subseteq \mathrm{Fin}(\mu_f)$.

On the other hand, if

$$\mu_f(A) = \sum_{i=1}^{\infty} f(i)\chi_A(i) < +\infty$$

then for any $\varepsilon > 0$ there is large enough n such that

$$\mu_f(A \setminus [1,n]) = \sum_{i>n} f(i)\chi_A(i) < \varepsilon,$$

meaning that $\|A\|_\varphi = 0$. □

Erdös-Ulam ideals

A function $f\colon \mathbb{N}\to\mathbb{R}^+$ is an *Erdös–Ulam function* if

$$\mu_f(\mathbb{N})=+\infty \qquad\text{and}\qquad \lim_{n\to\infty}\frac{f(n)}{\mu_f([1,n])}=0.$$

The condition in the definition of EU function also appeared in [MMŠT] when studying Darboux property of weighted densities. It is useful to notice that it can be stated in a different way: [MMŠT, Proposition 1.1]

Lemma 3. *Let $f\colon \mathbb{N}\to\mathbb{R}^+$ be a function such that $\mu_f(\mathbb{N})=+\infty$. Then*

$$\lim_{n\to\infty}\frac{f(n)}{\mu_f([1,n])}=0 \qquad\Leftrightarrow\qquad \lim_{n\to\infty}\frac{\max_{k\le n}f(k)}{\mu_f([1,n])}=0$$

Proof. Since $f(n)\le\max_{k\le n}f(k)$, the implication $\boxed{\Leftarrow}$ is trivial.

$\boxed{\Rightarrow}$ If the sequence $(f(n))$ is bounded, then both limits are equal to zero. (We have $\lim\limits_{n\to\infty}\mu_f([1,n])=\mu(\mathbb{N})=+\infty$.) So it suffices to check the case when this sequence is unbounded.

Let us choose for each n some k_n such that

$$f(k_n)=\max_{k\le n}f(k).$$

If $(f(n))$ is unbounded we have $k_n\to\infty$ for $n\to\infty$.

Now

$$\frac{\max_{k\le n}f(k)}{\mu_f([1,n])}=\frac{f(k_n)}{\mu_f([1,n])}\le\frac{f(k_n)}{\mu_f([1,k_n])}$$

and since the sequence on the RHS tends to zero for $n\to\infty$, the same is true for the sequence on the LHS. □

We are interested in the *Erdös–Ulam ideal*

$$\mathcal{EU}_f=\{A\subseteq\mathbb{N};\lim_{n\to\infty}\frac{\mu_f(A\cap[1,n])}{\mu_f([1,n])}=0\}.$$

This is precisely the ideal corresponding to the weighted density given by the function f.

We can define

$$\varphi_f(A)=\sup_n\frac{\mu_f(A\cap[1,n])}{\mu_f([1,n])}.$$

We can see that φ_f is a lsc submeasure.[2]

Proposition 6. *Let $f\colon \mathbb{N}\to\mathbb{R}^+$ be a function such that $\mu_f(\mathbb{N})=+\infty$ and $A\subseteq\mathbb{N}$. Then*

$$\mathcal{EU}_f=\operatorname{Exh}(\varphi_f).$$

Hence $\mathcal{EU}_f$ is an $F_{\sigma\delta}$ P-ideal.

[3]

[2]It is easy to check that if φ is a lsc submeasure, then so is $\varphi(A\cap[1,n])$ for any fixed n. Positive multiple of lsc submeasure is again lsc submeasure. The same is true for finite sums and arbitrary suprema.

[3]Originally I thought than in this proof I need f to be EU function, which was one of the reasons for including Lemma 3. However, the in the proof it suffices to assume that $\mu_f(\mathbb{N})=+\infty$.

Proof. We want to show that

$$\lim_{k\to\infty} \frac{\mu_f(A\cap[1,k])}{\mu_f([1,k])} = 0 \qquad \Leftrightarrow \qquad \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{\mu_f((A\setminus[1,n])\cap[1,k])}{\mu_f([1,k])} = 0$$

$\boxed{\Rightarrow}$ Fix $\varepsilon > 0$. Then there is an n_0 such that for $k > n_0$ we have

$$\frac{\mu_f(A\cap[1,k])}{\mu_f([1,k])} < \varepsilon.$$

Now this implies that for *arbitrary* k we get

$$\frac{\mu_f((A\setminus[1,n_0])\cap[1,k])}{\mu_f([1,k])} < \varepsilon.$$

Just notice that the above expression is zero for $k < n_0$. And for $k \geq n_0$ it can be estimated from above by ε, since $\mu_f((A\setminus[1,n_0])\cap[1,k])/\mu_f([1,k]) \leq \mu_f(A\cap[1,k])/\mu_f([1,k]) < \varepsilon$.

So we get

$$\sup_{k\in\mathbb{N}} \frac{\mu_f((A\setminus[1,n])\cap[1,k])}{\mu_f([1,k])} \leq \sup_{k\in\mathbb{N}} \frac{\mu_f((A\setminus[1,n_0])\cap[1,k])}{\mu_f([1,k])} \leq \varepsilon$$

for any $n \geq n_0$.

$\boxed{\Leftarrow}$ Fix $\varepsilon > 0$. There is an n_0 such that for *every* $k \in \mathbb{N}$

$$\frac{\mu_f((A\setminus[1,n_0])\cap[1,k])}{\mu_f([1,k])} < \frac{\varepsilon}{2}.$$

For a fixed n_0 we get

$$\lim_{k\to\infty} \frac{\mu_f([1,n_0])}{\mu_f([1,k])} \leq n_0 \cdot \lim_{k\to\infty} \frac{\max_{i\leq n_0} f(i)}{\mu_f([1,k])} = 0,$$

since $\mu_f(\mathbb{N}) = \lim\limits_{k\to\infty} \mu_f([1,k]) = +\infty$. So there is k_0 such that for $k \geq k_0$ the inequality

$$\frac{\mu_f([1,n_0])}{\mu_f([1,k])} < \frac{\varepsilon}{2}$$

holds.

Using the fact that (from subadditivity and monotonicity of μ_f)

$$\mu_f(A) \leq \mu_f(A\setminus[1,n_0]) + \mu_f([1,n_0])$$

we can combine the above two inequalities to get

$$\frac{\mu_f(A\cap[1,k])}{\mu_f([1,k])} < \varepsilon$$

for $k \geq k_0$. □

Analytic P-ideals

The following result can be found in [F, Theorem 1.2.5]. It shows that the ideals obtained in this way from lsc submeasures cover a very large class of naturally defined ideals.

Theorem 1 (Mazur, Solecki)**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then*

a) *$\mathcal{I}$ is an F_σ ideal iff $\mathcal{I} = \mathrm{Fin}(\varphi)$ for som e lsc submeasure φ.*

b) *$\mathcal{I}$ is an analytic P-ideal iff $\mathcal{I} = \mathrm{Exh}(\varphi)$ for som e lsc submeasure φ.*

c) *$\mathcal{I}$ is an F_σ P-ideal iff $\mathcal{I} = \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$ for som lsc submeasure φ.*

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