Pecherskii's Theorem

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Steinitz's theorem

$f \in X^*$ is *convergence functional* $\Leftrightarrow \sum_{k=1}^{\infty} |f(x_k)| < \infty$

Γ =set of all convergence functionals of the series $\sum_{n=1}^{\infty} x_n$

$\Gamma^{\perp} = \{x \in X \colon f(x) = 0 \text{ for all } f \in \Gamma\}$

Theorem (Steinitz's theorem)

Let $\sum_{k=1}^{\infty} x_k$ be a convergent series in an m-dimensional space E, and let $\sum_{k=1}^{\infty} x_k = s$. Then the sum range of the series is the affine subspace $s + \Gamma^{\perp}$, where $\Gamma^{\perp}$ is the annihilator of the set of convergence functionals: $\mathrm{SR}(\sum_{k=1}^{\infty} x_k) = s + \Gamma^{\perp}$.

Fonf's Lemma

[F, Lemma 2]:

Lemma

Let C be a convex set in a Banach space X, Γ a subset of X^* and $\Gamma^\perp$ its annihilator. If for every $f \notin \Gamma$ and every $T > 0$ there exist $x', x'' \in C$ such that $f(x') > T$ and $f(x'') < -T$, then $x \in C$ implies $x + \Gamma^\perp \subset C$.

$$\mathcal{Q} = \left\{ \sum_{i=1}^{n} \lambda_i x_i ; 0 \leq \lambda_i \leq 1; N = 1, 2, \dots \right\}$$

Lemma

Let X be and arbitrary Banach space, and let $\sum_{k=1}^\infty x_k$ be a conditionally convergent series in X. Then for any $x \in \overline{\mathcal{Q}}$ the set $x + \Gamma^\perp$ is contained in $\overline{\mathcal{Q}}$.

First step

$s' \in s + \Gamma^{\perp} \subseteq \overline{\mathcal{Q}}$
$\|s' - \sum_{i=1}^{n} \lambda_i x_i\| < \varepsilon_1$
$\|\sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \theta_i x_i\| < m \cdot \max_{k \geq 1} \|x_k\|$ (Rounding-off lemma)
$S_1 := \{i = 1, \ldots, n; \theta_i = 1\} \cup \{1\}$
$\|s' - s_1\| \leq \varepsilon_1 + m \cdot \max_{k \geq 1} + \|x_1\|$
Repeating the same procedure
$\|s' - s_n\| \leq \varepsilon_n + m \cdot \max_{k \geq n} + \|x_n\|$
we obtain a permutation such that a subsequence of partial sums converges to s'
$\lim_{j \to \infty} \left\|s' - \sum_{k=1}^{n_j} x_{\pi(k)}\right\| = 0$

Second step

Rearrangement lemma:

$$\left\| \sum_{i=1}^{k} x_{\pi(i)} \right\| \leq m \cdot \max_i \|x_i\| + (m+1) \left\| \sum_{i=1}^{n} x_i \right\| \qquad (1)$$

$$\lim_{j\to\infty} \left\| s' - \sum_{k=1}^{n_j} x_{\pi(k)} \right\| = 0 \qquad \Rightarrow \qquad \lim_{j\to\infty} \left\| \sum_{k=n_j+1}^{n_{j+1}} x_{\pi(k)} \right\| = 0$$

Reformulation

Let $\sum_{n=1}^{\infty} x_n$ in a Banach space X fulfills:

A. $(\forall \varepsilon > 0)$ $(\exists N = N(\varepsilon))$ $(\exists \delta > 0)$ $\{y_i\}_{i=1}^{n} \subset \{x_i\}_{i=N}^{\infty}$, $\|\sum_{i=1}^{n} y_i\| \leq \delta$, $\Rightarrow$ there exists a permutation π with

$$\max_{j \leq n} \left\| \sum_{i=1}^{j} y_{\pi(j)} \right\| \leq \varepsilon;$$

B. $(\forall \varepsilon > 0)$ $(\exists M = M(\varepsilon))$ $\{y_i\}_{i=1}^{n} \subset \{x_i\}_{i=M}^{\infty}$, $0 \leq \lambda_i \leq 1$, $i = 1, \ldots, n$, $\Rightarrow$ there exists $\{\theta_i\}_{i=1}^{n}$, $\theta_i \in \{0, 1\}$, for which

$$\left\| \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \theta_i y_i \right\| \leq \varepsilon.$$

Then $\mathrm{SR}(\sum_{n=1}^{\infty} x_n) = x + \Gamma^{\perp}$ holds for any $x \in \mathrm{SR}(\sum_{n=1}^{\infty} x_n)$.

Pecherskii's Theorem

$\sum_{k=1}^{\infty} x_k$ is *perfectly divergent* $\Leftrightarrow \sum \alpha_i x_i$ diverges for any $\alpha_i \in \{\pm 1\}^{\mathbb{N}}$.

Theorem

Let X be an arbitrary Banach space, let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series in X, and let $\sum_{k=1}^{\infty} x_k = x_0$. Further, suppose that no rearrangement makes the series perfectly divergent. Then $\mathrm{SR}(\sum_{k=1}^{\infty} x_k)$ is the closed affine subspace $x_0 + \Gamma^{\perp}$, where $\Gamma^{\perp}$ is the annihilator in X of the set $\Gamma \subset X^*$ of convergence functionals.

$$\mathrm{SR}(\sum_{k=1}^{\infty} x_k) = x_0 + \Gamma^{\perp}$$

Lemma 2.3.2

Lemma (2.3.2)

Suppose that there is no rearrangement that makes the series $\sum_{n=1}^{\infty} x_n$ perfectly divergent. Then for any $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that, for any finite collection, written in arbitrary order, of elements $\{y_1, y_2, \dots, y_n\}$ of the set $\{x_N, x_{N+1}, x_{N+2}, \dots\}$ one can find a collection of signs $\alpha_i = \pm 1$ for which

$$\max_{j \leq n} \left\| \sum_{i=1}^{j} \alpha_i y_i \right\| \leq \varepsilon.$$

Lemma 2.3.3 (Property B)

Lemma (2.3.3)

Let $\varepsilon > 0$ let $\{x_i\}_{i=1}^N$ be a set of elements of the normed space X, with the property that for any finite subset $\{y_i\}_{i=1}^m \subset \{x_i\}_{i=1}^N$ there exist signs $\alpha_i = \pm 1$ such that $\|\sum_{i=1}^m \alpha_i y_i\| \leq \varepsilon$. Then for any set of coefficients $\{\lambda_i\}_{i=1}^N$, $0 \leq \lambda_i \leq 1$ there exist "rounded off" coefficients $\theta_i \in \{0, 1\}$ for which

$$\left\| \sum_{i=1}^N \lambda_i x_i - \sum_{i=1}^N \theta_i x_i \right\| \leq \varepsilon.$$

Lemma 2.3.4–Chobanyan's lemma

Lemma (Chobanyan's lemma)

Let $\{x_i\}_{i=1}^n$ be elements of the space X, with $\sum_{i=1}^n x_i = 0$. Then there exists a permutation σ such that, for any choice of signs $\alpha_i = \pm 1$,

$$\max_{k\leq n} \left\| \sum_{i=1}^{k} \alpha_i x_{\sigma(i)} \right\| \geq \max_{k\leq n} \left\| \sum_{i=1}^{k} x_{\sigma(i)} \right\| . \tag{2}$$

Lemma 2.3.5

Lemma (2.3.5)

Let $\{x_i\}_{i=1}^n \subset X$, $\|\sum_{i=1}^n x_i\| \leq \varepsilon$, *and assume that for any permutation* ν *there exists a choice of signs* $\alpha_i = \pm 1$ *for which*

$$\max_{k \leq n} \left\| \sum_{i=1}^{k} \alpha_i x_{\nu(i)} \right\| \leq \varepsilon.$$

Then there exists a permutation σ *such that* $\max_{k \leq n} \|\sum_{i=1}^k x_{\sigma(i)}\| \leq 3\varepsilon$.

Modification of permutation π

$$\begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & n & n+1 \\ \pi(1) & \dots & \pi(k-1) & n+1 & \pi(k+1) & \dots & \pi(n) & \pi(n+1) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & n & n+1 \\ \pi(1) & \dots & \pi(k-1) & \cancel{n+1} & \pi(k+1) & \dots & \pi(n) & \pi(n+1) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & n-1 & n \\ \pi(1) & \dots & \pi(k-1) & \pi(k+1) & \pi(k+2) & \dots & \pi(n) & \pi(n+1) \end{pmatrix}$$

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