Hölder inequality and averages

Let us denote

$$A_p = \left(\frac{\sum_{k=1}^{n} a_k^p}{n} \right)^{\frac{1}{p}}.$$

I.e. A_p is a generalization of arithmetic average.

Proposition 1. $1 \leq p \leq q \Rightarrow A_p(\overline{a}) \leq A_q(\overline{a})$

Proof. The inequality

$$\left(\frac{\sum_{k=1}^{n} a_k^p}{n} \right)^{\frac{1}{p}} \leq \left(\frac{\sum_{k=1}^{n} a_k^q}{n} \right)^{\frac{1}{q}}$$

is equivalent to

$$\frac{\sum_{k=1}^{n} a_k^p}{n} \leq \left(\frac{\sum_{k=1}^{n} a_k^q}{n} \right)^{\frac{p}{q}}.$$

Let $b_k := a_k^p$ and $u = \frac{q}{p}$. Then $u \geq 1$ and we can rewrite this inequality as

$$\frac{\sum_{k=1}^{n} b_k}{n} \leq \left(\frac{\sum_{k=1}^{n} b_k^u}{n} \right)^{\frac{1}{u}}$$

$$\sum_{k=1}^{n} b_k \leq n^{(1-\frac{1}{u})} \left(\sum_{k=1}^{n} b_k^u \right)^{\frac{1}{u}}$$

The last inequality follows from Hölder inequality (for vectors $(1, 1, \ldots, 1)$ and $(b_1, \ldots, b_n)$). $\square$