`nets.tex` November 6, 2007

`http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/trf/`

In these notes I have put together some basic material concerning various notions of convergence in topological spaces. The results mentioned here are rather basic and they are considered standard, therefore I haven't included references for all of them. (I can put them into this text later if necessary.) Much of the material can be found in most standard topology textbooks, such as [E, Chapter 1.6], [Dug, Chapter X], [K, Chapter 2], [W, Chapter 4]. They are also sometimes included as the preliminaries in analytic texts, e.g. [AB, Chapters 2.4–2.6], [Dud, Chapter 2.1], [Pe, Section 1.3], [Z, p.134]. (This fact suggest that they are really important for applications.) See also [WIK, PLA]. Only the material on prime spaces is my addition.

We will see that all considered definitions of convergence are but various methods to describe the same thing. Of course, each of them might have its own advantages and can be used to describe different aspects more easily or to make them more apparent.

These notes have been used at our seminar when studying the paper [LD1].

Side effect of compiling this text from several sources is that the notation is not always consistent. E.g. we denote a directed set sometimes by Σ and sometimes by D. Probably several other similar problems can be found in the text.

I would like to thank V. Toma for correcting some errors (wrong definition of a subnet in the original version and several other inaccuracies) and for pointing out the connection with the closed limits (see Remark 1.1.15).

1 Convergence in topological spaces

There are two types of convergence, which are defined using filters (or, dually, via ideals). In the first of them we take a filter on a topological space and define a limit of this filter. The second (more general) possibility is that we take a filter on some index set I and a map $I \to X$. To avoid confusion, when we will speak about convergence of we will have a filter on a topological space X in mind. When dealing with a more general situation (a map from an index set to X), we will speak about ideal convergence instead.

1.1 Convergence of nets

We first describe the notion of convergence of nets (sometimes also called Moore-Smith sequences).

Definition 1.1.1. We say that $(D, \leq)$ is a *directed set,* if $\leq$ is a relation on D such that

(i) $x \leq y \wedge y \leq z \Rightarrow x \leq z$ for each $x, y, z \in D$;

(ii) $x \leq x$ for each $x \in D$;

(iii) for each $x, y \in D$ there exist $z \in D$ with $x \leq z$ and $y \leq z$.

In the other words a directed set is a set with a relation which is reflexive, transitive (=preorder or quasi-order) and upwards-directed.

The following two notions will be often useful for us

Definition 1.1.2. A subset A of set D directed by $\leq$ is *cofinal* in D if for every $d \in D$ there exists an $a \in A$ such that $d \leq a$.

A subset A of a directed set D is called *residual* if there is some $d_0 \in D$ such that $d \geq d_0$ implies $d \in A$.

Definition 1.1.3. A *net* in a topological space X is a map from any non-empty directed set Σ to X. It is denoted by $(x_\sigma)_{\sigma\in\Sigma}$.

The convergence of nets is defined analogously to the usual notion of convergence of sequences.

Definition 1.1.4. Let $(x_\sigma)_{\sigma\in\Sigma}$ be a net in a topological space X is said to be *convergent* to $x \in X$ if for each neighborhood U of x there exists $\sigma_0 \in \Sigma$ such that $x_\sigma \in U$ for each $\sigma \geq \sigma_0$.

$$(\forall_{U\in\mathcal{T}} U \ni x)(\exists\sigma_0 \in D)(\forall\sigma \geq \sigma_0)x_\sigma \in U \qquad (1) \quad \{\mathtt{EQ1}\}$$

If a net $(x_\sigma)_{\sigma\in\Sigma}$ converges to x, the point x is called a *limit* of this net.

The set of all limits of a net is denoted $\lim x_\sigma$.

The property (1) characterizing the convergence of net is sometimes called x_σ *is eventually in* U or *residually in* U, e.g. [K, Pe, W]. In the other words, the set of all σ's with $x_\sigma \in U$ is residual in Σ.

In general a limit of a net need not be unique (see Theorem 1.1.8). That's why we say $x \in \lim x_\sigma$ instead of $x = \lim x_\sigma$.

The following theorems shows that the convergence of nets describes completely the topology of X (and, consequently, it also characterizes continuity).

Theorem 1.1.5. *A point x belongs to $\overline{A}$ if and only if there exists a net consisting of elements of A which converges to x.*

Theorem 1.1.6. *A subset V of a topological space X is closed if and only if for each net $(x_\sigma)_{\sigma\in\Sigma}$ such that $x_\sigma \in V$ for each $\sigma \in \Sigma$, every limit of $(x_\sigma)_{\sigma\in\Sigma}$ belongs to V as well.*

Theorem 1.1.7. *Let X, Y be topological spaces. A map $f\colon X \to Y$ is continuous if and only if, whenever a net x_σ converges to x, the net $f(x_\sigma)$ converges to $f(x)$.*

Several important notions, such as Hausdorffness and compactness, can be characterized with the help of nets.

Theorem 1.1.8. *A topological space X Hausdorff $\Leftrightarrow$ every net in X has at most one limit.*

We say that the net $(y_e)_{e\in E}$ is *finer* than the net $(x_d)_{d\in D}$ or *subnet* of $(x_d)_{d\in D}$, if there exists a function φ of E to D with the following properties:

(i) For every $d_0 \in D$ there exists an $e_0 \in E$ such that $\varphi(e) \geq d_0$ whenever $e \geq e_0$.

(ii) $x_{\varphi(e)} = y_e$ for $e \in E$.

This definition can be formulated equivalently using the notion of cofinal map.[1]

Definition 1.1.9. [[Ru, Definition 3.3.13]] A function $f\colon P \to D$ from a preordered set to a directed set is *cofinal* if for each $d_0 \in D$ there exists $p_0 \in P$ such that $f(p) \geq d_0$ whenever $p \geq p_0$.

Hence a net $\sigma'\colon \Sigma' \to X$ is a subnet of a net $\sigma\colon \Sigma \to X$ if there exists a cofinal map $f\colon \Sigma' \to \Sigma$ with $\sigma' = \sigma \circ f$.

[1] Originally I had here an incorrect definition defining cofinal map as a map such that the range of f is cofinal. (The mistake was notices by V. Toma.) This definition is not equivalent to the definition given bellow. An easy counterexample is a map from $\omega + \omega$ to ω with the usual ordering given by $n \mapsto n$ and $\omega + n \mapsto 0$.

Let us note that some authors also require the function f to be non-decreasing. [2]

The notion of subnet is different from the notion subsequence, e.g. $1, 1, 2, 3, \dots$ is a subnet of $1, 2, 3, 4, \dots$. But it is precisely the notion needed to obtain the characterization of cluster points in a similar way as for sequences and subsequences. [G] contains characterization of all subnets of a sequence.

Definition 1.1.10. A point x is called a *cluster point of a net* $S = \{x_\sigma, \sigma \in \Sigma\}$ if for every $\sigma_0 \in \Sigma$ there exists a $\sigma \geq \sigma_0$ such that $x_\sigma \in U$.

$$(\forall_{U \in \mathcal{T}} U \ni x)(\forall \sigma_0 \in D)(\exists \sigma \geq \sigma_0) x_\sigma \in U \tag{2}$$ {EQ2}

The condition (2) is called (x_σ) is *frequently in* U or *cofinally in* U.

Theorem 1.1.11. *A point x is a cluster point of a net $(x_\sigma)_{\sigma \in \Sigma}$ if and only if there exists a subnet $(x_{\sigma'})_{\sigma' \in \Sigma'}$ of this net such that x is a limit of $(x_{\sigma'})$.*

Theorem 1.1.12. *A topological space X is compact $\Leftrightarrow$ every net in X has a cluster point.*

We also define a notion of ultranet which is the counterpart of the notion of ultrafilter.

Definition 1.1.13. A net (x_σ) in X is called *ultranet* if for every non-empty subset A of X the net is either eventually in A or eventually in $X \setminus A$. The terms universal or maximal net are also used.

Theorem 1.1.14. *Every net (x_σ) in X has a universal subnet. Any universal net converges to each of its cluster points.* (*I.e., if it has a cluster point, it converges.*)

We can note that, for any map $f \colon X \to Y$ the image of a universal net in X is again a universal net in Y.

Examples

- Sequences = nets on $(\mathbb{N}, \leq)$;
- Double sequences with the convergence in Pringsheim's sense = nets on $\mathbb{N} \times \mathbb{N}$ with the order $(x, y) \leq (x', y') \Leftrightarrow x \leq x' \wedge y \leq y'$ (i. e., $(\mathbb{N} \times \mathbb{N}, \leq \times \leq)$);
- Definition of Riemann integral;

Remark 1.1.15. Sometimes the notion of limit of a net of closed subsets of a topological space is defined as follows (see [Be, Definition 5.2.1]):[3]

If $(A_d)_{d \in D}$ is a net of subsets of X then

(i) The *lower closed limit* $\operatorname{Li} A_d$ of (A_d) consist of all such points x that each neighborhood of x intersects A_d for all d in some residual subset of A.

(ii) The *upper closed limit* $\operatorname{Ls} A_d$ of (A_d) consist of all such points x that each neighborhood of x intersects A_d for all d in some cofinal subset of A.

(iii) If $\operatorname{Li} A_d = \operatorname{Ls} A_d$ then (A_d) is said to be *Kuratowski-Painlevé convergent.*

Note that if we take $A_d = \{x_d\}$ then $\operatorname{Li}\{x_d\}$ is precisely the set of all limits of (x_d) and $\operatorname{Ls} A_d$ is precisely the set of all cluster points of (x_d). What can be considered an advantage of this notation is that with $\lim x_\sigma$ one usually associates a point, whereas $\operatorname{Li} A_s$ is always a set.

[2] [W, Definition 11.2]; [Be, p.149]; note that [W] has a different definition of cofinal map – as the map such that its range is cofinal – but he only uses cofinal monotone maps, which are cofinal in our sense as well.

[3] In [Be] this notion is defined only for Hausdorff spaces. I am not sure whether this condition is important.

1.2 Convergence of filters on X

Another common possibility used when dealing with the convergence in a topological space X is to consider filters on X.

Definition 1.2.1. A *filter* on a set X is a subset $\mathcal{F}$ of $\mathcal{P}(X)$ such that

(i) $\emptyset \notin \mathcal{F}$;

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(iii) $A \in \mathcal{F} \wedge A \subset B \Rightarrow B \in \mathcal{F}$.

Remark 1.2.2. Let us note that it is possible to define a filter in a subfamily $\mathcal{R}$ of $\mathcal{P}(X)$ which has largest element. In this case, we need to reformulate the second part of condition (iii) as $A \subset B \in \mathcal{R}$. Hence it is possible to define a filter in the family closed sets of X, which is used in the definition of Wallman compactification of a T_1-space.

We will deal with filters in $\mathcal{P}(X)$ only.

The following example of a filter will be of particular importance for us.

Example 1.2.3. Let X be a topological space. A *neighborhood filter* $\mathcal{N}(x)$ of a point $x \in X$ is the set of all neighborhoods of x. (Neighborhood of x is any subset V of X such there exists an open set U with $x \in U \subset V$.)

Definition 1.2.4. A filter on a topological space X is said to be *convergent* to a point $x \in X$ (or x is called a *limit* of $\mathcal{F}$) is every neighborhood of x belongs to $\mathcal{F}$. The set of all limits of $\mathcal{F}$ is denoted by $\lim \mathcal{F}$.

We see that $\mathcal{F} \to x \Leftrightarrow \mathcal{N}(x) \subseteq \mathcal{F}$.

Definition 1.2.5. A *filter-base* is a non-empty family $\mathcal{G} \subset \mathcal{P}(X)$ such that $\emptyset \notin \mathcal{G}$ and

(i) If $A_1, A_2 \in \mathcal{G}$, then there exists an $A_3 \in \mathcal{G}$ such that $A_3 \subset A_1 \cap A_2$.

If $\mathcal{G}$ is a filter-base then

$$\mathcal{F}_{\mathcal{G}} = \{A \subseteq X; (\exists G \in \mathcal{G}) A \supseteq G\}$$

is a filter on X.

A filter-base $\mathcal{G}$ converges to $x \in X \Leftrightarrow \mathcal{F}_{\mathcal{G}}$ converges to $x \Leftrightarrow$ every neighborhood U of x contains a member of $\mathcal{G}$.

For any map $f \colon X \to Y$ it was easy to get from a net in X a net in Y. In the case of filters the filter-bases do the work.

Lemma 1.2.6. *If $\mathcal{F}$ is a filter on X and $f \colon X \to Y$ is any function then $f(\mathcal{F}) = \{f[A]; A \in F\}$ is a filter-base in Y.*

Now we are ready to formulate the filter counterparts of the basic results we mentioned in connection with nets.

Theorem 1.2.7. *The point x belongs to $\overline{A}$ if and only if there exists a filter-base consisting of subsets of A converging to x.*

Theorem 1.2.8. *A mapping f of a topological space X to a topological space Y is continuous if and only if for every filter-base $\mathcal{G}$ in the space X and the filter-base $f(\mathcal{G}) = \{f[A] : A \in \mathcal{G}\}$ in the space Y we have*

$$f[\lim \mathcal{G}] \subset \lim f(\mathcal{G}).$$

Theorem 1.2.9. *A topological space X Hausdorff $\Leftrightarrow$ every filter in X has at most one limit.*

Definition 1.2.10. A point x is called a *cluster point of a filter* $\mathcal{F}$ if x belongs to closure of every member of $\mathcal{F}$.

Definition 1.2.11. We say that a filter $\mathcal{F}'$ is *finer* than a filter $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$.

Theorem 1.2.12. *A point x is a cluster point of a filter $\mathcal{F}$ if and only if there exists a filter $\mathcal{F}'$ which is finer than $\mathcal{F}$ and x is a limit of $\mathcal{F}$.*

Definition 1.2.13. A filter $\mathcal{F}$ is called an *ultrafilter* if for each $A \subseteq X$

$$(A \in \mathcal{F}) \ \vee \ (X \setminus A \in \mathcal{F})$$

Theorem 1.2.14. *If $\mathcal{F}$ is an ultrafilter and x is a cluster point of $\mathcal{F}$, then x is a limit of $\mathcal{F}$.*

Clearly, for any point $x \in X$ the system

$$\mathcal{F} = \{A \subseteq X; A \ni x\}$$

is an ultrafilter on X. It is called *principal ultrafilter.*

To obtain the existence of ultrafilters which are not principal (so called *free ultrafilters*) we need some form of Axiom of Choice. (A filter $\mathcal{F}$ is said to be *free* if the intersection of members of $\mathcal{F}$ is empty.)

Definition 1.2.15. A *centered system* on a set X is any system $\mathcal{C} = \{A_i; i \in I\}$ of subsets of X such that every finite intersection of sets from $\mathcal{C}$ is non-empty.

Using Zorn lemma one can show the following useful result

Theorem 1.2.16. *Every centered system is contained in an ultrafilter.*

The following theorem is often employed in the proof of Tychonoff's theorem.

Theorem 1.2.17. *A topological space X is compact $\Leftrightarrow$ every filter (every filter-base) in X has a cluster point.*

1.3 Equivalence of the notions of convergence of nets and convergence of filters

Theorem 1.3.1 ([E, Theorem 1.6.12]). *For every net $S = (x_\sigma)_{\sigma\in\Sigma}$ in a topological space X, the family $\mathcal{F}(S)$, consisting of all sets $A \subset X$ with the property that there exists a $\sigma_0 \in \Sigma$ such that $x_\sigma \in A$ whenever $\sigma \geq \sigma_0$, is a filter in the space X and*

$$\lim \mathcal{F}(S) = \lim S.$$

If a net S' is finer than the net S, then the filter $\mathcal{F}(S')$ is finer than the filter $\mathcal{F}(S)$.

(We see that the filter corresponding to a net (x_σ) is the set of all subsets A of X such that (x_σ) is eventually in A.)

Theorem 1.3.2 ([E, Theorem 1.6.13]). *Let $\mathcal{F}$ be a filter in a topological space X; let us denote by Σ the set of all pairs (x, A), where $x \in A \in \mathcal{F}$ and let us define that $(x_1, A_1) \leq (x_2, A_2)$ if $A_2 \subset A_1$. The set Σ is directed by $\leq$, and for the net $S(\mathcal{F}) = (x_\sigma)_{\sigma\in\Sigma}$, where $x_\sigma = x$ for $\sigma = (x, A) \in \Sigma$, we have $\mathcal{F} = \mathcal{F}(S(\mathcal{F}))$ and*

$$\lim S(\mathcal{F}) = \lim \mathcal{F}.$$

Using the above correspondence we obtain an ultranet from an ultrafilter and vice-versa.

Note that $S(\mathcal{F}(S)) \neq S$, since these two nets are defined on different directed sets. [4]

[4] Question: Is $S(\mathcal{F}(S))$ at least a subnet of S?

1.4 Ideal convergence

The dual notion of a filter is an ideal.

Definition 1.4.1. A subset $\mathcal{I}$ of $\mathcal{P}(X)$ is an *ideal* on X if

(i) $X \notin \mathcal{I}$;

(ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

(iii) $A \in \mathcal{I} \wedge A \supset B \Rightarrow B \in \mathcal{I}$.

Clearly, for each ideal $\mathcal{I}$ the system

$$\mathcal{F}(\mathcal{I}) = \{X \setminus A; A \in \mathcal{I}\}$$

is a filter. It is called the *dual filter* to $\mathcal{I}$. In a similar way we can obtain filter from ideal and it is obvious how to transfer various notions (convergence, cluster points, ...) between filters and ideals.

We recall Bourbakis's definition of a limit of a function. (This notion is due to Henri Cartan, although it seems to be reinvented several times by other authors.)

Definition 1.4.2 ([Bo, p.70]). Let f be a mapping of a set X into a topological space Y and let $\mathcal{F}$ be a filter on X. A point $y \in Y$ is said to be a *limit point*(or simply a *limit*) [resp. *cluster point*] of f with respect to filter $\mathcal{F}$ if y is a limit point (resp. cluster point) of the filter base $f[\mathcal{F}]$.

As we have already said, in order to distinguish the notions, which have similar names, we will use the name *ideal convergence* in this case.

Note that this definition can be reformulated as follows

Definition 1.4.3. Let Y be a topological space and $f\colon X \to Y$ be any map. Let $\mathcal{I}$ be an ideal on X. A point $y \in Y$ is said to be *$\mathcal{I}$-limit of the function* f if for each neighborhood U of Y the set

$$A(U) = \{x \in X; f(x) \notin U\} \tag{3}$$ {EQAU}

belongs to $\mathcal{I}$.

(This fact can be shown easily by observing that $A = f^{-1}(X \setminus U)$. See [Bo, p.71, Proposition 7].)

This notion is a common generalization of the filter convergence in a topological space from the preceding section and of the $\mathcal{I}$-convergence of sequences. The convergence of nets can be also considered as a special case – we need to take the ideal $\mathcal{I}_0(D)$ we will define later (denoted by $\mathcal{I}_0$ in [LD1], Definition 2.1.1). The corresponding filter is called the *section filter* of a directed set in [Bo, p.60]. (Because it is derived from the filter base consisting of *sections* if the directed set D.) A net is convergent to x if and only if if it is $\mathcal{I}_0(D)$ convergent, when considered as a map.

It is shown in [Bo, p.74,Proposition 8] that y is a cluster point of f with respect to $\mathcal{F}$ if and only if there is a finer filter $\mathcal{F}'$ such that x is a limit of $\mathcal{F}'$.

It is perhaps good to notice that the notion of cluster point is the same as the notion of $\mathcal{I}$-cluster point[5]

Definition 1.4.4. Let $\mathcal{I}$ be an ideal on X and $f\colon X \to Y$. A point $y \in Y$ is called *$\mathcal{I}$-cluster point* of f if for any neighborhood U of y

$$\{x \in X : f(x) \in U\} \notin \mathcal{I}.$$

[5]This fact is reproved in [LD1, Theorem 8].

1.5 Relation between convergence of nets, ideal convergence and prime spaces

It is clear that we have a one-to-one correspondence between ideals and filters. Similarly, for each directed set we have the section filter. Here we show that we can assign to each filter a topological space having only one accumulation point which can help to describe the ideal convergence. Sometimes this topological viewpoint could be useful.

Definition 1.5.1. A topological space is called a *prime space,* if it has precisely one accumulation point.

The notion of prime space was defined [FR] where the sequential convergence (i.e., ideal convergence for the ideal of finite sets) was studied. Some authors do not use the name prime spaces, they speak simply about topological spaces having unique accumulation point instead, e.g [BM].

For any filter on a set A we can define a prime space $P(\mathcal{F})$ on the set $A \cup \{\infty\}$ where $\infty \notin A$ in the following way: All points other from ∞ are isolated. The neighborhoods of ∞ are the sets of the form $\{\infty\} \cup F$ where $F \in \mathcal{F}$.

This correspondence between filters and prime spaces can be found e.g. in [Č, Section 2] or [Do, Proposition 2,3].

Similarly, to an ideal $\mathcal{I}$ we can assign the dual filter $\mathcal{F}(\mathcal{I})$ and, consequently, the prime space $P(\mathcal{F}(\mathcal{I}))$. For the sake of brevity, we will use again the notation $P(\mathcal{I})$.

The reason why prime spaces are interesting for us is the following:

Lemma 1.5.2. *Let X be a topological space, $x \in X$, and $f\colon A \to X$. Let $\mathcal{I}$ be an ideal on A Let us define a map $\overline{f}\colon P(\mathcal{I}) \to X$ by $\overline{f}(a) = f(a)$ for $a \in A$ and $f(\infty) = x$. Then x is $\mathcal{I}$-limit of f if and only if $\overline{f}$ is continuous.*

The proof can be seen easily from (3), for the case of sequences it was shown e.g. in [Sl, Proposition 3.1].

Let $\mathcal{I}$ an on A. Let us denote $P := P(\mathcal{I})$. We can see immediately that for any $V \subseteq A$:
$V \in \mathcal{I} \Leftrightarrow V$ is closed in P.
$V \notin \mathcal{I} \Leftrightarrow \infty \in \overline{V}$.
$V \in \mathcal{F}(\mathcal{I}) \Leftrightarrow V \cup \{\infty\}$ is open.

We can assign a prime space to each directed set as well – from a directed set we arrive to the section filter $\mathcal{F}_0(D)$ and from this filter to the prime space $P(\mathcal{F}_0(D))$. We can denote it by $P(D)$. The underlying set of this space is $D \cup \{\infty\}$. The topology of $P(D)$ can be easily described using $\leq$. All points in D are isolated and $A \cup \{\infty\}$ is open if and only if A contains the set $D_a = \{d \in D; d \geq a\}$ for some $a \in D$. Again, we see that

Lemma 1.5.3. *A net $(x_d)_{d\in D}$ converges to X if and only if the map $f\colon P(D) \to X$ given by $f(\infty) = x$ and $f(d) = x_d$ is continuous.*

We could note that a map $f\colon E \to D$ between to directed set is cofinal if and only if the corresponding map $\overline{f}\colon P(E) \to P(D)$ between the prime spaces is continuous (see Lemma 4.1.2).

2 Notes on some results from [LD1]

2.1 Definitions and notations

Definition 2.1.1. The *section filter* $\mathcal{F}_0(D)$ of a directed set $(D, \leq)$ is formed by all residual subsets of D, i.e., for $A \subseteq D$ we have

$$A \in \mathcal{F}_0(D) \qquad \Leftrightarrow \qquad (\exists d_0)(d \geq d_0 \Rightarrow d \in A).$$

The dual ideal is denoted by $\mathcal{I}_0(D)$.

An ideal $\mathcal{I}$ on D is D-admissible $\Leftrightarrow$ it contains the ideal $\mathcal{I}_0(D)$. (We have seen the same situation in the case of the $\mathcal{I}$-convergence of double sequences [DKMW].)

Clearly, if $\mathcal{J} \subseteq \mathcal{I}$ then $\mathcal{J}\text{-}\lim f = x$ implies $\mathcal{I}\text{-}\lim f = x$. From this we have that every convergent net is $\mathcal{I}$-convergent if $\mathcal{I}$ is D-admissible.

Example 1: In this example, the directed set has a maximal element. If m is the maximal element of a directed set D, then any net $(x_d)_{d\in D}$ converges to x if and only if each neighborhood of x contains x_m.

2.2 Basic properties

Theorem 1 says that in a Hausdorff space $\mathcal{I}$-limit of a net is unique. In another words, it show that in a Hausdorff space $\mathcal{I}$-limits of functions are unique (since the proof does not use the fact that $\mathcal{I}$ is ideal on a directed set.) The converse (Theorem 2) follows from Theorem 1.1.8.

Theorem 3, which characterizes the closed set using $\mathcal{I}$-convergent nets, follows from Theorem 1.1.6. Similarly, Theorem 4 follows from Theorems 1.1.7 and 1.2.8. (The theorem is valid for functions $f\colon X \to Y$ – probably a typo.)

2.3 $\mathcal{I}$-cluster points

Theorem 6 claims that x is an $\mathcal{I}$-cluster point of a net D, then there is a subnet on a directed set E and an ideal $\mathcal{J}$ on E such that x is $\mathcal{J}$-limit of this subnet. In fact, we do not need to take a subnet, it suffices to take a finer ideal on the same directed set (see Theorem 1.2.12).

We can sketch the proof (with rewriting the proof to the language of ideals). Let x be a cluster point of $(x_\sigma)_{\sigma\in\Sigma}$. This means that for each neighborhood U of x_0 the set $B(U) := \{\sigma; x_\sigma \in U\}$ does not belong to $\mathcal{I}$. Let us define a new ideal

$$\mathcal{J} = \{(D \setminus B(U)) \cup I; U \in \mathcal{N}(x), I \in \mathcal{I}\}.$$

Note that $\mathcal{J}$ does not contain the set D. Indeed, $(D \setminus B(U)) \cup I = D$ would imply $I \supseteq B(U)$ and $B(U) \in \mathcal{I}$, a contradiction. Therefore $\mathcal{J}$ is indeed an ideal on D. The inclusion $\mathcal{I} \subseteq \mathcal{J}$ implies that $\mathcal{J}$ is D-admissible and $x_\sigma \xrightarrow{\mathcal{J}} x_0$ since each $D \setminus B(U) = \{x_\sigma; x_\sigma \notin U\}$ is in $\mathcal{J}$.

In the proof of Theorem 6 we can see that the authors have obtained in fact a stronger result. They have found a subnet which is convergent to x_0.

Theorem 2.3.1. *If $(s_n)_{n\in D}$ is a net and x_0 is an $\mathcal{I}$-cluster point of (s_n), then there is a subnet of (s_n) which converges to x_0.*

This follows from implications $\mathcal{I}$-cluster point $\Rightarrow$ cluster point (which holds for each admissible ideal) and cluster point $\Rightarrow$ convergent subnet (Theorem 1.1.11).

In Theorem 7 the authors have in fact shown that x_0 is an $\mathcal{I}_D$-limit, not only $\mathcal{I}_D$-cluster point:

Theorem 2.3.2. *Let $(t_e)_{e\in E}$ be a subset of a net $(s_n)_{n\in D}$ and let i be the corresponding cofinal mapping $i\colon E \to D$. If $(t_e)_{e\in E}$ is $\mathcal{I}$-convergent to x_0 then $(s_n)_{n\in D}$ is $\mathcal{J}$-convergent to x_0 for the ideal*

$$\mathcal{J} = \{A \subset D; i^{-1}(A) \in \mathcal{I}\}.$$

Proof. The convergence of (t_e) implies that, for each neighborhood U of x_0,

$$A = \{e; t_e = s_{i(e)} \notin U\} \in \mathcal{I}.$$

Let $A' = \{e; s_d \notin U\}$. Clearly, $i^{-1}(A') = \{e; t_e = s_{i(e)} \notin U\} = A \in \mathcal{I}$, which implies $A \in \mathcal{I}$. □

This result can be easily interpreted using prime spaces. It is easy to show, that $\mathcal{J}$ is precisely the ideal such that the topology of the corresponding prime space is quotient w.r.t. the map i (resp. the extension $\overline{i}$ of this map). A thus a map f from $P(\mathcal{J})$ to X is continuous if and only if $f \circ i$ is continuous.

Theorems 8, 9 - we are not using the fact that we are working with nets.

Theorem 9 is a restatement of [LD2, Theorem 10].

Theorem 9: completely separable = different name for second countable (countable basis)

2.4 $\mathcal{I}^*$-convergence and the condition (DP)

Definition of $\mathcal{I}^*$-convergence: Clearly, the authors mean that $M \subseteq D$ is directed by the relation $\leq$ restricted to M.

This definition includes the condition that M is such subset of D, which is again directed. This condition was (for me) quite difficult to understand. But it is easy to note, that if M is cofinal in D, then it fulfills this condition. In all the proofs of this section, only such M's were used.

3 Historical notes

3.1 Convergence of nets in topological spaces

Defined in [MS].

3.2 Filter convergence in topological spaces

The general definition of a convergence of a filter is due to Henri Cartan [Car3, Car2]. It was used as the basic notion of convergence in [Bo]. However, some authors mention that filter base was used by L. Vietoris [V] under the name "Kräntze" prior to H. Cartan ([Pr, p.46], [Re]). R. M. Dudley [Dud, p.76] mentions that Caratheodory used filter bases in [Car1] and M. H. Stone was in fact dealing with filters in [St].

Equivalence with nets: [Ba1, Ba2].

4 Further notes

4.1 Various definitions of subnets

Remark 4.1.1. We have seen that in fact two different definitions of a subnet appear in the literature. A subnet of a net $(x_d)_{d \in D}$ is a net $(x_{d'})_{d' \in D}$ such that there is a map $f\colon D' \to D$ such that $x_{f(d')} = x_{d'}$ for each $d' \in D'$ and f fulfills the condition

(i) $\forall d_0 \in D\ \exists d'_0 \in D'$ such that $d' \geq d'_0$ implies $f(d') \geq d_0$ (see [E, K]), i.e., f is cofinal,

(ii) f is cofinal and non-decreasing (from [W]).

These 2 conditions are not equivalent,[6] but they correspond in the sense that if we define a cluster point of a net as a limit of some of its subnets, then we obtain the same notion of the cluster point regardless of which of these 2 definitions of the subnet we use.

[6] TODO Example

Perhaps it would be good (i) since it looks a little bit counterintuitive at the first sight. As we will see this condition is in fact very natural if we look at it from a correct viewpoint.

Lemma 4.1.2. *Let $f\colon \Sigma' \to \Sigma$ be a map between two directed sets. The following conditions are equivalent.*

(i) *f is a cofinal map (see Definition 1.1),*

(ii) *the map $f\colon \Sigma' \to P(\Sigma)$ is a convergent net,*

(iii) *the extension $\overline{f}\colon P(\Sigma') \to P(\Sigma)$ is continuous,*

(iv) *$f^{-1}(M)$ is residual in Σ' whenever M is residual in Σ*

(v) *if M is cofinal in Σ' then $f[M]$ is cofinal in Σ*

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are clear (and we have mentioned them before)

4.2 Filter convergence of sequences

Is a special case of Cartan's notion of limit; this concept was rediscovered later several times by different authors. The novelty of the paper [KŠW] is considering $\mathcal{I}^*$-convergence besides the $\mathcal{I}$-convergence.

Questions

1. Clearly, any filter $\mathcal{F}$ can is directed as $(\mathcal{F}, \supseteq)$. (Similarly, for ideals.) Can the convergence of filters be reformulated as the convergence of a net on this directed set with elements in some space of subsets of the topological space X?
2. What would be the corresponding notion for $\mathcal{I}$-limit point and $\mathcal{I}^*$-convergence in the case of functions?
3. Is there some topological characterization of $\mathcal{I}$-cluster points (similar to the characterization of $\mathcal{I}$-limits via prime spaces)?
4. Topological characterization of $\mathcal{I}^*$-convergence? (Something similar to the characterization of $\mathcal{I}$-convergence via prime spaces.)
5. Does there exists for each filter $\mathcal{F}$ a directed set D such that $\mathcal{F} = \mathcal{F}(D)$ (i.e., $\mathcal{F}$ is the section filter of D). Characterization of section filters? (Maybe some of the results of [CP] could be interesting in the connection with this problem.)
6. We have seen in Lemma 4.1.2 that cofinal maps correspond to continuous maps between prime spaces. What concept for directed sets would correspond to quotient maps?
7. Perhaps some equivalent condition in Lemma 4.1.2 could be expressed using section filters.

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