Since we have been dealing with the problem of various definitions of subnet in literature, I have gathered here at least some of them.

Definitions

Subnet defined using cofinal map

Engelking [E]:

Definition 1. We say that the net $S' = \{x_{\sigma'}, \sigma' \in \Sigma'\}$ is *finer* than the net $S = \{x_\sigma, \sigma \in \Sigma\}$ it there exists a function φ of Σ' to Σ with following properties:

(i) For every $\sigma_0 \in \Sigma$ there exists a $\sigma'_0 \in \Sigma'$ such that $\varphi(\sigma') \geq \sigma_0$ whenever $\sigma' \geq \sigma'_0$.

(ii) $x_{\varphi(\sigma')} = x_{\sigma'}$ for $\sigma' \in \Sigma'$.

A point x is called a *cluster point of a net* $S = \{x_\sigma, \sigma \in \Sigma\}$ if for every $\sigma_0 \in \Sigma$ there exists a $\sigma \geq \sigma_0$ such that $x_\sigma \in U$.

[K] has the same definition, the only difference is using the term *subnet* instead of finer net of [E]. He also mentions that: "Notice that each cofinal subset E of D is directed by the same ordering, and that $\{S_n, n \in E\}$ is a subnet of S. (Let N be the identity function on E, and the second condition of the definition becomes the requirement that E be cofinal.) This is a standard way of constructing subnets, and it is unfortunate that this simple variety of subnet is not adequate for all purposes. (2.E)"

The same definition is also used in [AB, Definition 2.15], [P, 1.3.2], [Dud, p.48], [BvR, p.206].

Pedersen's book [P] contains also the following note: "In most cases we can choose h to be monotone, and then, in order to have a subnet, it suffices to check that for each λ in Λ, there is a μin M with $\lambda \leq h(\mu)$."

Dudley [Dud] uses the term *strict subnet* – I am not sure, but by "nonstrict subnet" he probably means a subnet given by a cofinal subset of directed set.

This definition is reformulated using the notion of cofinal map as follows:

Definition 2. [[R, Definition 3.3.13]] Let $\mathbb{A}$ and $\mathbb{B}$ be directed sets. A map $\phi\colon \mathbb{B} \to \mathbb{A}$ is called *cofinal* if, for each $\alpha \in \mathbb{A}$ there is $\beta_\alpha \in \mathbb{B}$ such that $\alpha \preceq \phi(\beta)$ for all $\beta \in \mathbb{B}$ such that $\beta_\alpha \preceq \beta$.

Definition 3. [[R, Definition 3.3.14]] Let S be a non-empty set, and let $(x_\alpha)_{\alpha\in\mathbb{A}}$ and $(y_\beta)_{\beta\in\mathbb{B}}$ be nets in S. Then $(y_\beta)_{\beta\in\mathbb{B}}$ is a subnet of $(x_\alpha)_{\alpha\in\mathbb{A}}$ if $y_\beta = x_{\phi(\beta)}$ for a cofinal map $\phi\colon \mathbb{B} \to \mathbb{A}$.

In [Gä] we can find still another definition using the section filter. Let J and I be directed sets, $\mathfrak{K}$ and $\mathfrak{L}$ be the corresponding filter. $(y_j)_J$ is said to be *subnet* (*Teilnetz*) of $(x_i)_i$ if there exists a map $f\colon J \to I$ such that $x_{f(j)} = y_j$ for each $ij \in J$ and $\mathfrak{K} \subseteq f(\mathfrak{L})$ (i.e., for each $K \in \mathfrak{K}$ there exists $L \in \mathfrak{L}$ with $f[L] \subseteq K$.)

Again, this is an equivalent definition to Definition 1. This shows (in my opininion) the connection between the notion of a subnet and the notion of finer filter.

Summary: Definition 1 (or an equivalent definition) in [AB, BvR, Dud, E, Gä, K, R, P]. (Cofinal map is mentioned explicitly only in [R].)

Subnet defined using increasing cofinal map

Willard [W, Definition 11.2]:

Definition 4. A *subnet* of a net $P\colon \Lambda \to X$ is the composition $P \circ \varphi$, where $\varphi\colon M \to \Lambda$ is an increasing cofinal function from a directed set M to Λ. That is,

(a) $\varphi(\mu_1) \leq \varphi(\mu_2)$ whenever $\mu_1 \leq \mu_2$ (φ is *increasing*),

(b) for each $\lambda \in \Lambda$, there is some $\mu \in M$ such that $\lambda \leq \varphi(\mu)$ (φ is *cofinal* in Λ).

Notice that (b) in the preceding definition says that φ has cofinal range, but for monotone maps this is equivalent to Definition 2.

Munkres [M, p.188]: $i \leq j \Rightarrow g(i) \leq g(j)$ and $g[K]$ is cofinal in J.

The same definition is in [B, p.149] and .

Summary: I found this definition only in [B, Ge, M, W]. (The notion of cofinal map is mentioned explicitly in [W], but the definitions in remaining books are equivalent to this one.)

Definitions of cofinal map

It is quite common when dealing with ordinals, that cofinal map is defined as a map with cofinal range. (This can be found in various set theoretic textbooks, as Tourlakis: Lectures in logic and set theory, vol. 2; Ciesielski: Set Theory for the Working Mathematician, probably also some other books.) But here we deal only with functions from ordinals to ordinals (i.e., linearly ordered sets) which is a different situation. And again in this case, the first thing these authors do is showing that if there is a cofinal increasing function from an ordinal α to a ordinal β, then there exists a cofinal order-preserving map.

Various names for upper section and cofinal subsets

section: terminal set in [Dug]

cofinal subset:

cofinal map:

section filter: zu $\preceq$ gehörige Filter in [Gä]

Are these definitions "compatible"?

Cluster points

TODO

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