http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/trf/pideals/

I've put together various notes concerning the notions I will mention in my talk today. Since they are from various areas, no title seemed to be most appropriate for me.

Descriptive set theory

We will use some facts from descriptive set theory, namely the notions of Borel and analytic sets and some facts on trees (König's lemma). References for DST: [Ke], [Mo], [Sr]. Also [A, Chapter 3] seems to be a good introduction into the problematic of Polish spaces, Borel and analytic sets.

Trees

We will only need the trees of countable height therefore we will follow the definition (and notation) of [Ke]. For a more general definition of a tree see e.g. [JW].

For any given set A we will work with finite sequences of elements of A. We denote by $A^{<\omega} = \bigcup\limits_{n\in\omega} A^n$ the set of all finite sequences of elements of A. Similarly, A^ω is the set of all (infinite) sequences of elements of A.

For any $x \in A^\omega$ we denote by $x|n$ the finite sequence $(x_0, \dots, x_{n-1})$.[1] We say that $s \in A^n$ is an *initial segment* of $x \in A^\omega$ if $s = x|n$. We will write $s \subset n$ is s is an initial segment of x.

Definition 1 ([Ke, Definition 2.1]). *Tree* = subset $T \subseteq A^{<\omega}$ closed under inital segments. (I.e., if $t \in T$ and $s \subset t$, then $s \in T$.)

An *infinite branch* of T is a sequence $x \in A^{\mathbb{N}}$ such that $x|n \in T$, for all n.

Height of a tree T is $\sup_{t\in T} ht(t)$ where $ht(t)$ is the length of the sequence t.

A tree T is called *finite splitting* if for every $s \in T$ there are at most finitely many $a \in A$ with $s\hat{\ }t \in T$.

Theorem 1 (König's lemma). *If T is a finitely splitting tree on a non-empty set A with height ω then it has an infinite branch.*

The assumption that T is finitely splitting cannot be omitted - just consider a tree consisting of countably many branches – for each $n \in \omega$ we take one branch of length n.

Borel sets and analytic sets

Descriptive set theory often deals with subsets of Polish spaces (=topological space which is homeomorphic to a complete metric space). Most frequently the Cantor space 2^ω and the Baire space ω^ω (considered as the product of discrete spaces) are employed. We will only work with the Cantor space 2^ω.

Borel sets are the sets from the smallest σ-algebra containing open sets.

Definition 2 ([Ke, Definition 14.1]). Let X be a Polish space. A set $A \subseteq X$ is called *analytic* if there exists a Polish space Y and a continuous function $f\colon Y \to X$ with $f[Y] = A$.

It can be shown that instead of arbitrary Polish space one can use the Cantor space (or the Baire space, or any fixed uncountable Polish space).

[1] This is the same as the restriction of x (as a function $\omega \to A$) to the subset $n \subseteq \omega$.

Cantor space

We will often work with the Cantor space 2^ω – the product of countably many copies of the 2-point discrete space. This space is useful since subsets of ω can be identified in an obvious way – as the characteristic sequences – with the points of this spaces and therefore we can study systems of subsets of ω (such as ideals, filters or topologies) as subsets of a topological space (study whether they are open, closed, Borel etc.).

On the examples of ideals we can see that many "natural" or "well-behaved" ideals are indeed Borel or analytic.

Therefore it could be useful to have a more detailed look at the topology of this space.

If we identify the points of this space with subsets of ω, the topology of Cantor space has a base consisting of sets

$$U_{F,G} = \{A \subseteq \omega; F \subseteq A, G \cap A = \emptyset\}$$

for F, G finite. The same topology is given by the subbase Subbase $S_a = \{A \subseteq \mathbb{N}; a \in A\}$ $S'_a = \{A \subseteq \mathbb{N}; a \notin A\}$ (any $a \in \omega$).

The basic sets mentioned above are clopen. Spaces having a base consisting of clopen sets are called *zero-dimensional.*

The topology of the Cantor cube is given by the metric

$$d(A, B) = 2^{-\Delta(A,B)},$$

where $\Delta(A, B)$ is the smallest integer contained the symmetric difference of A and B.

Cantor space can be embedded into the space $\langle 0, 1\rangle$ using the embedding

$$(a_n) \mapsto \sum_{n=1}^{\infty} \frac{2a^n}{3^n}$$

where $(a_n) \in \{0, 1\}\omega$ is a sequence of 0's and 1's.

We can view the point of this space as sequences of natural numbers. In this context the following base could be useful:

$$U_s = \{x \in 2^\omega; s \subset x\}$$

where $s \in 2^{<\omega}$ and $s \subset x$ means that the (finite) sequence s is an initial segment of x.

A neighborhood base of a sequence $b \in 2^\omega$ is given by the sets

$$B_n = \{x \in 2^\omega; b|n \subset x\}.$$

We can also formulate convergence and ideal convergence in this space in the language of subsets or sequences.

If X_k are subsets of ω then X_k converges to X if for each $n \in \omega$ there exists $k_0 \in \omega$ such that $k > k_0 \Rightarrow X_k \cap n = X \cap n$.

If x_k are sequences of natural numbers then x_k converges to x if for each $n \in \omega$ there exists $k_0 \in \omega$ such that $k > k_0 \Rightarrow x_k|n = x|n$.

Continuous images and closed subspaces

The paper [FMRS] uses several basic facts concerning uncountable compact metric spaces, the Cantor space and the unit interval. We will mention some of them.

Note that every compact metric space is complete (since every Cauchy sequence is bounded and in a compact metric space every bounded sequence has a convergent subsequence).

We first show that the Cantor space can be embedded into every uncountable compact metric space. Recall that a *perfect space* is a space having no isolated points. A *perfect set* in a topological space is a set which contains all its accumulation points.

Theorem 2. *If X is an infinite perfect complete metric space then in contains a copy of the Cantor space 2^ω.*

Sketch of the proof. We construct inductively for each sequence $s \in 2^\omega$ a decreasing sequence of open subsets such that their diameter tends to 0. Moreover at each level these sets are disjoint. By the completeness there is unique point in the intersection of every such decreasing sequence of sets. □

Theorem 3 (Cantor-Bendixson theorem). *Every complete metric space can be written as $X = P \cup C$ where P is a perfect set and C is a countable set.*

A point X in a topological space is called *condensation point* if every open neighborhood of x in uncountable.

In the proof the set P is the set of all condensation points of X.

Consequently, in each uncountable compact metric space there exists an uncountable perfect subspace, which, in turn contains a copy of the Cantor space.

Next we observe that every compact metric space is a continuous image of a closed subspace of the Cantor space.

Theorem 4. $I = \langle 0, 1\rangle$ *is a continuous image of the Cantor space.*

Proof. The continuous map is

$$(a_n) \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

□

Every compact metric space is a X subspace of I^ω. (Since every compact space is first-countable and every compact space can be embedded into $I^{w(X)}$ by Tychonoff's theorem.) We have a continuous map $f\colon (2^\omega)^\omega \to I^\omega$ (the power of the continuous map from the above theorem). Clearly, $(2^\omega)^\omega$ is homeomorphic to the Cantor space 2^ω. Thus the restriction of f from $f^{-1}(X)$ to X is a continuous map from a closed subspace of the Cantor space to X.

Let us note that a stronger theorem can be shown: Every non-empty compact metrizable spaces is a continuous images of the Cantor space 2^ω. [Ke, Theorem 4.18]

The notion of a limit along a filter

We have discussed several times about the question where the notion of the convergence along a filter was defined for the first time and should be credited for the invention of this notion.

The notion of limit along a filter on ω should not be confused with the notion of the limit of a filter in a topological space.

Let X be a topological space and $\mathcal{F}$ be a filter on X. A point x is called a *limit of a filter* $\mathcal{F}$ if every neighborhood of x is a member of $\mathcal{F}$.

The convergence of filters corresponds in a certain sense with the convergence of nets (or Moore-Smith sequences), which include convergence of sequences as a special case. But the filter which corresponds to the sequence is a filter on X rather than on ω. For more details on the convergence of filters in topological spaces see e.g. [E]. Here this notion is attributed to Bourbaki, Topológie generale, 1940.

Nevertheless, the notion of convergence along a filter (in the dual sense to our $\mathcal{I}$-convergence) seems to be relatively old as well. In [FMRS] this notion is attributed to Katětov [Kat] and for the case of ultrafilters to A. R. Bernstein [Be].

Limit along an ultrafilter is used frequently in order to show the existence of a Banach limit. (This approach is preferred in set-theoretical expositions, whereas analysts usually employ Hahn-Banach theorem in this proof.) We can find this proof in several standard set-theoretical textbooks as [BŠ, p.122, Definition 8.23] or [HJ, p.206, Definition 2.7]. (The former is the book where I have seen this notion for the first time. It is used in this textbook as an illustration of usefulness of ultrafilters - the authors provide here a construction of Banach limit and of finitely additive measure on integers extending the asymptotic density. The main tool is the limit along an ultrafilter.) The construction of Banach limit via ultrafilter on ω is credited in some textbooks to A. Robinson [R].

The limit along an ultrafilter is used in [GS], too (the authors refer to [Be]).

Hence the oldest use of limit along an ultrafilter I was able to track down was the paper [R], in [Kat] the author is not restricted to the ultrafilters only.

At last (with Juraj Činčura) we were able to find out that the notion of a limit along a filter (or at least its more general version) was indeed defined by Bourbakists. We can find there [Bo2, p.68, Definition 1] or [Bo1, p.99, Opredelenie 3] (both of them are reprints/translations of the 1940 book):

Definition 3. Let $f\colon X \to Y$ be a map, where X is a set and Y is a topological space. Let $\mathcal{F}$ be a filter on X. A point $y \in Y$ is said to be a *limit point* (or simply *a limit*) of $\mathcal{F}$ if $\mathcal{F}$ is finer than the neighborhood filter $\mathcal{B}(x)$ is finer than X.

Equivalent condition: Filter base $f[\mathcal{F}]$ converges to y (in the topological space X).

The notion of a convergence in this way encapsulates both, filter converges in topological spaces and the convergence of a sequence along a filter, as special cases.

Products of filters

The authors of [FMRS] make use of *Fubini product* of two ideals defined as

$$\mathcal{I} \times \mathcal{J} = \{A \subseteq \omega \times \omega; \{n \in \omega; A_n \notin \mathcal{J}\} \in \mathcal{I}\} \tag{1}$$ {FUBPROD}

where $A_n = \{m \in \omega; (n,m) \in A\}$.

There exist some other types of products of ideals. We can quote the following two definitions from [CN, p.156] (for the case of filters).

$$\mathcal{F} \cdot \mathcal{G} = \{A \subseteq \omega \times \omega; \{n \in \omega; A_n \in \mathcal{G}\} \in \mathcal{F}\} \tag{2}$$ {FILTCDOT}

$$\mathcal{F} \times \mathcal{G} = \{C \subseteq \omega \times \omega; C \supseteq A \times B \text{ for some } A \in \mathcal{F}, B \in \mathcal{G}\} \tag{3}$$ {FILTTIMES}

The filter $\mathcal{F} \times \mathcal{G}$ is called the *product* of $\mathcal{F}$ and $\mathcal{G}$. (The above definition says, in the other words, that it is given by a base $\{A \times B; A \in \mathcal{F}, B \in \mathcal{G}\}$.)

The dual ideal to $\mathcal{F} \cdot \mathcal{G}$ looks like this

$$\mathcal{I} \cdot \mathcal{J} = \{A \subseteq \omega \times \omega; \{n \in \omega; A_n \in \mathcal{F}(\mathcal{J})\} \in \mathcal{I}\} \tag{4}$$ {IDCDOT}

Let us denote by $\mathcal{F} \times_F \mathcal{G}$ the result of dual construction of the Fubini product of ideals. (This filter contains the sets such that $\{n \in \omega; \omega \setminus A_n \in \mathcal{G}\} \in \mathcal{F}$.)

Lemma 1. *For any two filters the inclusion $\mathcal{F} \times \mathcal{G} \subset \mathcal{F} \cdot \mathcal{G}$ holds.*

Proof. If $C \supseteq A \times B$ with $A \in \mathcal{F}$ and $B \in \mathcal{G}$, then we have $B \subseteq C_n$ and $C_n \in \mathcal{G}$ for each $n \in A$, thus $A \subseteq \{n \in \omega; A_n \in \mathcal{G}\}$ and $\{n \in \omega; A_n \in \mathcal{G}\} \in \mathcal{F}$. □

Let us denote by $\mathcal{F} \times_F \mathcal{G}$ the result of dual construction of the Fubini product of ideals. (This filter contains the sets such that $\{n \in \omega; \omega \setminus A_n \in \mathcal{G}\} \in \mathcal{F}$.)

Lemma 2. *For any two filters the inclusion $\mathcal{F} \cdot \mathcal{G} \subset \mathcal{F} \times_F \mathcal{G}$ holds.*

Proof. Follows from the fact that $A_n \in \mathcal{F}(\mathcal{I}) \Rightarrow A_n \notin \mathcal{I}$. $\square$

Thus we have

$$\mathcal{F} \times \mathcal{G} \subset \mathcal{F} \cdot \mathcal{G} \subset \mathcal{F} \times_F \mathcal{G}.$$

If $\mathcal{F}$ is the filter of cofinite sets then $\mathcal{F} \times \mathcal{F} = \mathcal{F} \cdot \mathcal{F}$ consists of all sets such that for all but finitely many n's the vertical cuts $A - n$ are cofinite.

The Fubini product $\mathcal{F} \times_F \mathcal{G}$ consists of all sets such that for all but finitely many n's the vertical cuts A_n are infinite.

Ideals, filters and prime spaces

It is clear that we have a one-to-one correspondence between ideals an filters. Here we show that we can assign to each filter a topological space having only one accumulation point which can help to describe the ideal convergence. Sometimes this topological viewpoint could be useful.

Definition 4. A topological space is called a *prime space,* if it has precisely one accumulation point.

The notion of prime space was defined [FR] where the sequential convergence (i.e., ideal convergence for the ideal of finite sets) was studied. Some authors do not use the name prime spaces, they speak simply about topological spaces having unique accumulation point instead, e.g [BM].

For any filter on a set A we can define a prime space $P(\mathcal{F})$ on the set $A \cup \{\infty\}$ where $\infty \notin A$ in the following way: All points other from ∞ are isolated. The neighborhoods of ∞ are the sets of the form $\{\infty\} \cup F$ where $F \in \mathcal{F}$.

This correspondence between filters and prime spaces can be found e.g. in [Č, Section 2] or [D, Proposition 2,3].

We only work with filters on ω, so we will use the point ω instead of ∞. If we will assign a prime spaces to an ideal, we will use the same notation $P(\mathcal{I})$. (It should be clear from the context whether we work with an ideal or with a filter.)

The reason why prime spaces are interesting for us is the following:

Lemma 3. *Let X be a topological space, $x \in X$, $x_n \in X$ for each $n \in \omega$. Let $\mathcal{I}$ be an ideal on ω Let us define a map $f\colon P(\mathcal{I}) \to X$ by $f(n) = x_n$ and $f(\omega) = x$. Then $\mathcal{I}\text{-}\lim x_n = x$ if and only if f is continuous.*

For the detailed proof see e.g. [Sl1, Proposition 3.1].

It could be useful to express some topological notions for the space $P(\mathcal{F})$ in terms of subsets of ω and some notions concerning the $\mathcal{I}$-convergence in the language of topology (a kind of “dictionary”).

Let $\mathcal{I}$ be the dual ideal to a filter $\mathcal{F}$. Let us denote $P := P(\mathcal{F})$. We can see immediately that for any $A \subseteq \omega$:
$A \in \mathcal{I} \Leftrightarrow A$ is closed in P.
$A \notin \mathcal{I} \Leftrightarrow \omega \in \overline{A}$.
$A \in \mathcal{F}(\mathcal{I}) \Leftrightarrow A \cup \{\omega\}$ is open.

$(x_n) \restriction A$ is $\mathcal{I} \restriction A$-convergent $\Leftrightarrow$ the restriction $f|_{A\cup\{\omega\}}$ is continuous. (Here f is the map from the above lemma.)

An ideal $\mathcal{I}$ has BW $\Leftrightarrow$ every map $f\colon P \to I$ has a continuous restriction to $A \cup \{\omega\}$ for some dense subset A of P.

The product $\mathcal{F} \times \mathcal{G}$ defined in (3) corresponds to a subspace of the topological product $P(\mathcal{F}) \times P(\mathcal{G})$.

The product $\mathcal{F} \cdot \mathcal{G}$ corresponds to a subspace of A-sum defined in [Sl2]. This construction was inspired by the sequential sum from [AF] and [FR]. It is also similar to the brush of [Kan].

If we keep in mind that the sets from $\mathcal{I}$ are closed, then the definition of the Rudin-Keisler order reminds the definition of the quotient map. Indeed, $\mathcal{I} \leq_{RK} \mathcal{J}$ if and only if there exits a quotient map $P(\mathcal{J}) \to P(\mathcal{I})$ which maps only ω to ω (in the other words, $f^{-1}(\{\omega\}) = \{\omega\}$).

Analytic ideals

Submeasures

Recall that by identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is F_σ if it is an F_σ subset of the Cantor space; $\mathcal{I}$ is analytic if it is a continuous image of a G_δ subset of the Cantor space.

References for submeasures and analytic ideals: [F3], [F2], [So1], [So2]...

Definition 5. *Submeasure* on ω (=monotonic+subadditive): $\phi\colon \mathcal{P}(\omega) \to \langle 0, +\infty\rangle$

$$\begin{gathered}
\phi(\emptyset) = 0 \\
A \subseteq B \;\Rightarrow\; \phi(A) \leq \phi(B) \\
\phi(A \cup B) \leq \phi(A) + \phi(B) \\
\phi(\{n\}) < +\infty \text{ for } n \in \omega
\end{gathered}$$

(Some authors do not include the last condition into the definition of the submeasure. If we want to work only with submeasures such that the ideal $\mathrm{Fin}(\phi)$ defined below is admissible, we need this condition.)

Lower semicontinuous

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap n).$$

For any lower semicontinuous submeasure we define

$$\|A\|_\phi = \limsup_{n\to\infty} \phi(A \setminus n) = \lim_{n\to\infty} \phi(A \setminus n),$$

where the second inequality follows by the monotonicity of ϕ. (It is denoted by $\phi^\infty(A)$ in [F3].)[2]

$\|A\|_\phi$ is a submeasure as well (subadditive, monotone).

Subaddivity: $\|A \cup B\|_\phi \leq \limsup(\phi(A \setminus n) + \phi(B \setminus n)) \leq \limsup \phi(A \setminus n) + \limsup \phi(B \setminus n) = \|A\|_\phi + \|B\|_\phi$

For any $f\colon \omega \to [0, +\infty)$

$$\nu_f(A) = \sum_{i\in A} f(i)$$

is a lsc submeasure.

[2]TODO Is this equality true? Try to find some examples of lower semicontinuous submeasures.

For any submeasure we can define the following two ideals (although we will study them mostly for lsc submeasures):

$$\mathrm{Exh}(\phi) = \{A \subseteq \mathbb{N}; \|A\|_\phi = 0\}$$
$$\mathrm{Fin}(\phi) = \{A \subseteq \mathbb{N}; \phi(A) < \infty\}$$

Lsc submeasure is called *finite* if $\phi(\omega) < \infty$ ($\Leftrightarrow \mathrm{Fin}(\phi) = \mathcal{P}(\omega)$.)
Lsc submeasure is *exhaustive* if $\mathrm{Exh}(\phi) = \mathrm{Fin}(\phi)$.
Ideals of the form $\mathrm{Fin}(\phi)$ are implicit in Mazur's paper [Maz].

Relation between $\mathrm{Exh}(\phi)$ *and* $\mathrm{Fin}(\phi)$.

We show that $\mathrm{Exh}(\phi) \subseteq \mathrm{Fin}(\phi)$ for any lsc submeasure. By the last condition in the definition of submeasure we have $\phi(A) < +\infty$ for any finite set. Now let $A \in \mathrm{Exh}(\phi)$, i.e., $\lim\limits_{n\to\infty} \phi(A \setminus n) = 0$. Then there exists an n_0 such that $\phi(A \setminus n_0) < \infty$. Thus

$$\phi(A) \leq \phi(A \cap n_0) + \phi(A \setminus n_0) < \infty.$$

(The first summand is finite because the set $A \cap n_0$ is finite.)

The following example shows that the opposite inclusion is not in general true:
Let us define ϕ by $\phi(A) = 1$ for $A \neq \emptyset$. Then ϕ is a lsc submeasure and
$\mathrm{Exh}(\phi) = \mathrm{Fin}$
$\mathrm{Fin}(\phi) = \mathcal{P}(\omega)$
thus $\mathrm{Exh}(\phi) \subsetneq \mathrm{Fin}(\phi)$.

The following example shows that we need the assumption $\phi(\{n\}) < +\infty$ in the proof of $\mathrm{Exh}(\phi) \subseteq \mathrm{Fin}(\phi)$
Now let $\phi(A) = \infty$ if $1 \in A$ and 0 otherwise. This is a lsc submeasure.
$\mathrm{Exh}(\phi) = \mathcal{P}(\omega)$
$\mathrm{Fin}(\phi) = \{A \subseteq \omega; 1 \notin A\}$.
In this case we have $\mathrm{Fin}(\phi) \subsetneq \mathrm{Exh}(\phi)$.

Theorem 5 ([So2]). *TFAE*

(i) $\mathcal{I}$ *is an analytic P-ideal*

(ii) $\mathcal{I} = \mathrm{Exh}(\phi)$ *for some lower semicontinuous submeasure* ϕ *on* ω

Therefore every analytic P-ideal is automatically $F_{\sigma\delta}$.

Theorem 6 ([Maz]). *TFAE*

(i) $\mathcal{I}$ *is an* F_σ *ideal*

(ii) $\mathcal{I} = \mathrm{Fin}(\phi)$ *for some lower semicontinuous submeasure* ϕ *on* ω

Theorem 7 ([F2, Theorem 1.2.5],[So2]). *TFAE*

(i) $\mathcal{I}$ *is an* F_σ *P-ideal*

(ii) $\mathcal{I} = \mathrm{Fin}(\phi) = \mathrm{Exh}(\phi)$ *for some lower semicontinuous submeasure* ϕ *on* ω

The authors of [FMRS] use the negation of the following property to characterize analytic P-ideals with the property BW:
If $\varepsilon > 0$ then there exists a partition $\omega = A_1 \cup \ldots \cup A_s$ with $\phi(A_i) < \varepsilon$.

Such measures are called *compact* in [P] or [ŠV]. It is called a *diffuse submeasure* by some other authors ([F1], [FS], [FZ]). In [Mar] the term *strongly subatomic* is used for a finitely additive measure with similar properties.

Various classes of analytic ideals

Summable ideals

If $\phi = \nu_f$ for some f, then $\mathrm{Fin}(\phi) = \{A \subseteq \omega; \nu_f(A) < +\infty\}$ is called a *summable ideal.* Every summable ideal is a P-ideal.

Erdös-Ulam ideals

$f\colon \omega \to [0,+\infty)$ such that $\sum_{i=0}^\infty f(i) = +\infty$ and

$$\limsup \frac{f(n)}{\sum_{i\in n} f(i)} = \limsup \frac{\nu_f(A\cap n)}{\nu_f(n)} = 0$$

$$\mathcal{I}_f = \left\{ A : \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i\in n} f(i)} = 0 \right\}$$

Ideals $\mathcal{I}_d$ and $\mathcal{I}_\delta$ are EU-ideals.
Every EU-ideal is P-ideal.

Density ideals

If I_n are pairwise disjoint intervals in ω and μ_n is a measure on I_n. Then

$$\mathcal{Z}_\mu = \{A \subseteq \omega; \limsup \mu_n(A\cap I_n) = 0\}$$

is called a *density ideal.*

For $\phi = \sup_n \mu_n$ we have $\mathcal{Z}_\mu = \mathrm{Exh}(\phi)$, hence every density ideal is a P-ideal.

$$\mathcal{I}_d = \{A \subseteq \omega; \limsup_n 2^{-n}|A\cap I_n| = 0\},$$

where $I_n = [2^n, 2^{n+1}) \Rightarrow \mathcal{I}_d$ is a density ideal. (A detailed proof of a similar identity for $\mathcal{I}_d$ can be found in [C].) More generally: every EU-ideal is a density ideal.

Generalized density ideals

Let $\{I_n; n\in\omega\}$ be a partition of ω into finite intervals and ϕ_n be a submeasure on I_n for every n. Assume moreover that $\limsup at^+(\phi_n) = 0$, where $at^+(\phi) = \sup\{\phi(\{i\})\}$. Then

$$\mathcal{Z}_\phi = \{A; \limsup \phi_n(A\cap I_n) = 0\}$$

is called a *generalized density ideal.*

Generalized density ideals are known to be P-ideals.

LV ideals are another large class of ideals. The definition is not included in [FMRS] and it is only sketched in [F3]. They form a subclass of the class of generalized density ideals.

Summary

EU-ideal $\Rightarrow$ density ideal $\Rightarrow$ generalized density ideal $\Rightarrow$ analytic P-ideal
summable ideal $\Rightarrow$ F_σ P-ideal

$\mathcal{I}$-small sets

A set $A \subseteq \omega$ is $\mathcal{I}$-small if there are sets A_s (with $s \in 2^\omega$) such that for all s we have:

(i) $A_\emptyset = A$,

(ii) $A_s = A_{s\frown 0} \cup A_{s\frown 1}$,

(iii) $A_{s\frown 0} \cap A_{s\frown 1} = \emptyset$ and

(iv) $(\forall b \in 2^\omega)$ $(\forall X \subseteq \omega)$ $(\forall n \in \omega)$ $X \setminus A_{b\restriction n} \in \mathcal{I} \Rightarrow X \in \mathcal{I}$.

It is known that the system $S_{\mathcal{I}}$ of all $\mathcal{I}$-small sets forms an ideal an $\mathcal{I} \subseteq S_{\mathcal{I}}$. They were introduced in [F3]. We include here also some results from this paper:
If $\mathcal{I}$ is a density ideal then there is an $\mathcal{I}$-positive set $A \in \mathcal{S}_{\mathcal{I}}$. (Moreover such set is contained in every $\mathcal{I}$ positive set.)
If $\mathcal{I}$ is a LV-ideal then $\mathcal{S}_{\mathcal{I}} = \mathcal{I}$.
If $\mathcal{I}$ is a density ideal such that $\sup \mu_n(I_n) = +\infty$, then $\mathcal{S}_{\mathcal{I}}$ is a proper F_σ ideal properly including $\mathcal{I}$.

Orderings

Apart from the inclusion one can define several other partial orders on the set of ideals on ω.

Rudin-Keisler order. $\mathcal{I} \leq_{RB} \mathcal{J}$ if there exists a function $f\colon \omega \to \omega$ such that $A \in \mathcal{I}$ if and only if $f^{-1}(A) \in \mathcal{J}$.

We can list here a few simple examples.

Fubini product: $\mathcal{I} \leq_{RK} \mathcal{I} \times \mathcal{J}$, $\mathcal{J} \leq_{RK} \mathcal{I} \times \mathcal{J}$ (the map f is the function which a pair (m, n) to m or n respectively).

Rudin-Blass order is defined in the same way as $\leq_{RK}$, but the function f is required to be finite-to-one.

Small uncountable cardinals

Recently several cardinal numbers, defined some combinatorial properties, with the property $\aleph_1 \leq \varkappa \leq \mathfrak{c}$ have been used in various branches in mathematics. Some of them were generalized to cardinal numbers of Boolean algebras.

Here we mention only a few well-known small uncountable cardinals. Much more can be found in the extensive literature on this topic. As the basic references we can mention [vD], [V].

$\mathcal{P}(X)$ denotes the power set of X, $[X]^\omega$ the set of all countably infinite subsets of X and $[X]^{<\omega}$ the set of all finite subsets of X, ${}^\omega\omega$ the set of all functions from ω into ω.

For a family $\mathcal{F} \subset [\omega]^\omega$, we say that $\mathcal{F}$ has the *strong finite intersection property* provided every finite subfamily has an infinite intersection, and an infinite set A is called a *pseudointersection* of $\mathcal{F}$ provided $A \subset^* F$ for all $F \in \mathcal{F}$.

We define the *mod finite order* $\leq^*$ on ${}^\omega\omega$ as follows: $f \leq^* g$ provided that there exists $N \in \omega$ such $f(n) \leq g(n)$ holds that for all $n \geq N$.

A set $X \subset {}^\omega\omega$ is *dominating* (in the mod finite order) if for every $f \in {}^\omega\omega$ there exists $g \in X$ such that $f \leq^* g$, and X is *bounded* (in the mod finite order) if there exists $g \in {}^\omega\omega$ such that $f \leq^* g$ for all $f \in X$.

Definitions of small cardinals. $\mathfrak{a} = \min\{|A|; A \subset [\omega]^\omega$ is an infinite, maximal almost disjoint family in $\omega\}$.
$\mathfrak{b} = \min\{|B|; B \subset {}^\omega\omega$ is unbounded in the mod finite order $\}$.
$\mathfrak{d} = \min\{|D|; D \subset {}^\omega\omega$ is dominating in the mod finite order $\}$.
$\mathfrak{s} = \min\{|S|; S \subset [\omega]^\omega$ is a splitting family on $\omega\}$.
$\mathfrak{p} = \min\{|P|; P \subset [\omega]^\omega$ has the strong finite intersection property but no $X \in [\omega]^\omega$ is a pseudointersection for $P\}$.
$\mathfrak{t} = \min\{|T|; T \subset [\omega]^\omega$ is a tower on $\omega\}$.
$\mathfrak{i} = \min\{|I|; I \subset [\omega]^\omega$ is a maximal independent family on $\omega\}$.
$\mathfrak{u} = \min\{|U| U \subset [\omega]^\omega$ is a base for an ultrafilter on $\omega\}$.

A family $\mathcal{T} \subset [\omega]^\omega$ is a (decreasing) *tower* provided there exist an ordinal α and a bijection $f\colon \alpha \to \mathcal{T}$ such that $\beta < \gamma < \alpha$ implies that $f(\gamma) \subset^* f(\beta)$, and no infinite set A is a pseudointersection of $\mathcal{T}$.

Basic inequalities $\aleph_1 \le \mathfrak{p} \le \mathfrak{t} \le \mathfrak{b} \le \mathfrak{d} \le \mathfrak{i} \le \mathfrak{c}$

$\mathfrak{t} \le \mathfrak{s} \le \mathfrak{d}$

$\mathfrak{b} \le \mathfrak{a} \le \mathfrak{c}$

For the proofs see [vD, Theorem 3.1].

$cf(\mathfrak{t}) = \mathfrak{t} \le cf(\mathfrak{b}) = \mathfrak{b} \le cf(\mathfrak{d}) \le \mathfrak{d}$

Of course, under CH all these cardinals are equal.

Cardinal $\mathfrak{p}$. A centered family with no infinite pseudo-intersection is called *power* by some authors. [T]

MA $\Rightarrow \mathfrak{p} = \mathfrak{c}$

E.g., in [Mi] in fact only $\mathfrak{p} = \mathfrak{c}$ was need to construct an ideal with the T-property, so not the full strength of CH was necessary.

Splitting number. A family $\mathcal{A} \subseteq [\omega]^\omega$ is a *splitting family* if for any $B \in [\omega]^\omega$ there exists $A \in \mathcal{A}$ such that $\operatorname{card} A \cap B = \operatorname{card}(\omega \setminus A) \cap B = \aleph_0$.

We denote by $\mathfrak{s}$ the smallest size of a splitting family on ω.

$\aleph_1 \le \mathfrak{s} \le \mathfrak{c}$

[JW, p.80] MA+$\neg$CH $\Rightarrow \mathfrak{s} > \aleph_1$

Independence number. A family $\mathcal{I} \subset [\omega]^\omega$ is an *independent family* provided for every $A, B \in [\mathcal{I}]^{<\omega}$ if $A \neq \emptyset$ and $A \cap B = \emptyset$ then $\bigcap A \setminus \bigcup B \neq \emptyset$.

$\mathfrak{i} = \min\{|I|; I \subset [\omega]^\omega \text{ is a maximal independent family}\}$

$\aleph_1 \le \mathfrak{s} \le \mathfrak{d} \le \mathfrak{i} \le \mathfrak{c}$

Questions

1. Examples of: a non-analytic ideal; an analytic P-ideal which is not F_σ ... Is every F_σ ideal analytic? Is Fin $\times$ Fin analytic?

The ideal $\mathcal{I}_d$ is an analytic P-ideal (since it is EU-ideal). If it was an F_σ ideal, then by [FMRS, Proposition 3.4] it would have hFinBW property. But we know that $F_\sigma \notin BW$.

2. Is it possible to modify the definition of $\mathcal{S}_\mathcal{I}$ in a such way that $A \in \mathcal{S}'_\mathcal{I} \Leftrightarrow \mathcal{I} \restriction A \notin$ FinBW?

3. Does the following hold: $\mathcal{I} \restriction A \le_{RK} \mathcal{J}$ for any $A \notin \mathcal{I}$?

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