Notes about uniform spaces from Preuss, ...

Definition. Uniformity ω *is a filter on* $X \times X$ *such that:*
U1) $W \in \omega \Rightarrow \triangle = \{(x,x) : x \in X\} \subset W$
U2) $W \in \omega \Rightarrow W^{-1} \in \omega$ *U3)* $W \in \omega \Rightarrow (\exists W^* \in \omega)(W^*)^2 \subset W$
(X, ω) - uniform space
Elements of ω - entourages
For each $V \in \omega$ $A \subset X$ *is called* V*-small if* $A \times A \subset V$.

If (X, d) is a metric (pseudometric) space and $V_\varepsilon = \{(x,y) : d(x,y) < \varepsilon\}$ for each $\varepsilon > 0$, then $B = \{V_\varepsilon; \varepsilon > 0\}$ is a base for uniformity ω_d on X.

Proclaim. *Let* (X, ω) *be a uniform space. For each* $x \in X$ *and each* $V \in \omega$ *let* $V(x) = \{y : (x,y) \in V\}$*. Then* $\mathcal{X}_\omega = \{O \subset X :$ *there is some* $V \in \omega$ *with* $V(x) \subset O\}$ *is a topology on* X.

Definition. *Let* (X, ω), (X', ω') *be a uniform spaces. A map* $f \colon X \to X'$ *is called* uniformly continuous *provided that one ofthe following two equivalent conditions is satisfied:*

(i) For each $W' \in \omega'$ *there is some* $W \in \omega$ *such that* $(x,y) \in W \Rightarrow (f(x), f(y) \in W'$.

(ii) $(f \times f)^{-1}[W'] \in \omega$ *for each* $W' \in \omega'$*, where* $f \times f : (x,y) \mapsto (f(x), f(y))$.

initial uniformity, product space, supremum of family of uniformities on X

Theorem. *For each uniformity* ω *on a set* X *there is a family* $(d_V)_{v \in \omega}$ *if pseudometrics on* X *such that for the corresponding uniformities* D_v *the following is true:* $\omega = \sup\{D_v : V \in \omega\}$.

Theorem. *A topological space* $(X, \mathcal{X})$ *is uniformizable (i.e. there is a uniformity* ω *on* X *such that* $X_\omega = \mathcal{X}$*) if and only if it is completely regular.*

Definition. *A uniform space* (X, ω) *is called* separated *iff* $\bigcup\limits_{W \in \omega} W = \triangle$.

A uniform space (X, ω) *is called* pseudometrizable *(metrizable) provided that there is a pseudometric (metric)* d *on* X *such that* $\omega_d = \omega$.

Theorem. *An uniform space is pseudometrizable if and only if it has countable base.*

An uniform space is metrizable if and only if it is separable and has countable base.

Definition. *A filter* $\mathcal{F}$ *on a uniform space* (X, ω) *is called a* Cauchy filter *provided that for each* $W \in \omega$ *there is some* $F \in \mathcal{F}$ *such that* $F \times F \subset W$.

A uniform space (X, ω) *is called* complete *provided that each Cauchy filter on* (X, ω) *is convergent (in* $(X, \mathcal{X}_\omega)$*).*

Theorem. *Let* (X, ω) *be a separated uniform space. Then there is a dense embedding into a complete separated uniform space. Moreover, it has properties of reflection and is determined uniquely. It is called* complete hull.

Definition. *A uniform space* (X, ω) *is called* totally bounded *provided that one of the following equivalent conditions is satisfied:*

(i) For each $V \in \omega$ there is a finite cover of X by V-small sets.

(ii) For each $W \in \omega$ there is a finite subset E of X such that $W[E] = X$, where $W[E] = \bigcup_{x \in E} W(x)$.

(iii) Each ultrafilter on X is Cauchy filter.

Theorem. *A uniform space (X, ω) is compact (i.e. $(X, \mathcal{X}_\omega)$ is compact) if and only if it is complete and totally bounded.*