MacLane: **Categories for Working Mathematician**

1 Categories, Functors, and Natural Transformations

1.1 Axioms for Categories

1.2 Categories

Discrete categories. A category is *discrete* when every arrow is an identity.

1.3 Functors

full=hom-function is surjective (for every pair of objects), faithfull=injective

1.4 Natural Transformations

Given two functors $S, T\colon C \to B$ a *natural transformation* $\tau : S \dot{\to} T$ is a function which assigns to each object c of C an arrow $\tau_c = \tau c\colon Sc \to Tc$ of B in such a way that every arrow $f\colon c \to c'$ in C yields a diagram

$$\begin{tikzcd} c \arrow[d, "f"] & Sc \arrow[r, "\tau c"] \arrow[d, "Sf"'] & Tc \arrow[d, "Tf"] \\ c' & Sc' \arrow[r, "\tau c'"] & Tc' \end{tikzcd}$$

which is commutative.

A natural transformation τ with every component τc invertible in B is called *natural equivalence* or better a *natural isomorphism.*

An *equivalence* between categories C and D is defined to be a pair of functors $S\colon C \to D$, $T\colon D \to C$ together with natural isomorphisms $I_c \cong T \circ S$, $I_D \cong S \circ T$.

1.5 Monics, Epis and Zeros

monic=monomorphism, epic=epimorphism

right inverse=section, left inverse=retraction

An object t is *terminal* in C if to each object a in C there is exactly one arrow $a \to t$. Dual: *initial* object.

A *null object* z in C is an object which is both initial and terminal. For any two objects a and b of C there is a unique arrow $a \to z \to b$, celled the *zero arrow* from a to b.

A *groupoid* is a category in which every arrow is invertible. A typical groupoid is the *fundamental groupoid* $\pi(X)$ of a topological space X. (objects=points, arrow $x \to x'$ =homotopy classes of paths from x to x') *connected groupoid:* there is an arrow joining any two of its objects. (It is determined up to isomorphism by a group.)

1.6 Foundations

1.7 Large Categories

Ab-categories ? (hom-sets are abelian groups)

2 Construction on categories

2.1 Duality

2.2 Contravariance and Opposites

2.3 Products of categories

Functors $S\colon B \times C \to D$ from a product category are called *bifunctors*.

Proposition 2.3.1. *Let B, C and D be categories. For all objects $c \in C$ and $b \in B$, let*

$$L_c\colon B \to D, \quad M_b\colon C \to D$$

be functors such that $M_b(c) = L_c(b)$ for all b and c. Then there exists a bifunctor $S\colon B \times C \to D$ with $S(-, c) = L_c$ for all c and $S(b, -) = M_b$ for all b if and only if for every pair of arrows $f\colon b \to b'$ and $g\colon c \to c'$ one has

$$M_{b'} g \circ L_c f = L_{c'} f \circ M_b g.$$

These equal arrow in D are then the value $S(f, g)$ of the arrow function of S.

Proposition 2.3.2. *For bifunctors S, S', the function α which assigns to each pair of objects $b \in B$, $c \in C$ an arrow $\alpha(b, c)\colon S(b, c) \to S'(b, c)$ is a natural transformation $\alpha : S \xrightarrow{\cdot} S'$ (i.e., of bifunctors) if and only if $\alpha(b, c)$ is natural in b for each $c \in C$ and natural in c for each $b \in B$.*

Such natural transformations appear in the fundamental definition of adjoint functor (Chapter IV.) A functor $F\colon X \to C$ is the *left adjoint* of a functor $G\colon C \to X$ (opposite direction) when there is a bijection

$$hom_C(Fx, c) \cong hom_x(x, Gc)$$

natural in $x \in X$ and $c \in C$.

2.4 Functor categories

$B^C - Funct(C, B)$ with objects the functors $T\colon C \to B$ and morphism the natural transformations. $Nat(S, T) := B^C(S, T) = \{\tau | \tau : S \xrightarrow{\cdot} T \text{ natural}\}$.

2.5 The Category of All Categories

We have defined a "vertical" composite $\tau \cdot \sigma$,

$$\begin{tikzcd} C \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2, "\downarrow\sigma"] \arrow[r, phantom, shift right=2, "\downarrow\tau"] & B \end{tikzcd}$$

Given functors and natural transformations,

$$\begin{tikzcd} C \arrow[r, shift left=2, "S"] \arrow[r, shift right=2, "T"'] \arrow[r, phantom, "\downarrow\tau"] & B \arrow[r, shift left=2, "S'"] \arrow[r, shift right=2, "T'"'] \arrow[r, phantom, "\downarrow\tau'"] & A \end{tikzcd} \tag{2.1}$$

one may construct natural transformation $\tau' \circ \tau$ as

$$(\tau' \circ \tau)c = T'\tau c \circ \tau' Sc = \tau' TC \circ S'\tau c. \tag{2.2}$$

This composition is readily shown to be associative. It moreover has identities. It is convenient to let the symbols S for a functor also denote the identity transformation $1_S : S \stackrel{\cdot}{\to} S$. The definition (2.2) can then be rewritten using also the vertical composition, as

$$\tau' \circ \tau = (T' \circ \tau) \cdot (\tau' \circ S) = (\tau' \circ T) \cdot (S' \circ \tau) \tag{2.3}$$

There is a more general rule. Given three categories and four transformations

$$\begin{tikzcd} C \arrow[r, bend left=40] \arrow[r] \arrow[r, bend right=40] \arrow[r, phantom, shift left=2.5, "\downarrow\sigma"] \arrow[r, phantom, shift right=2.5, "\downarrow\tau"] & B \arrow[r, bend left=40] \arrow[r] \arrow[r, bend right=40] \arrow[r, phantom, shift left=2.5, "\downarrow\sigma'"] \arrow[r, phantom, shift right=2.5, "\downarrow\tau'"] & A \end{tikzcd} \tag{2.4}$$

the "vertical" composites under $\cdot$ and the "horizontal" composites under $\circ$ are related by the identity *(interchange law)*

$$(\tau' \cdot \sigma') \circ (\tau \cdot \sigma) = (\tau' \circ \tau) \cdot (\sigma' \circ \sigma) \tag{2.5}$$

Exercise 4: Let G be a topological group with identity element e, while σ, σ', τ, τ' are continuous paths in G starting and ending at e. $\circ$ - skladanie ciest, $\cdot$ pointwise product. Then interchange law holds.

Exercise 5: (Hilton=Eckmann). Let S be a set with two (everywhere defined) binary operations $\cdot \colon S \times S \to S$, $\circ \colon S \times S \to S$ which both have the same (two-sided) unit element e and which satisfy the interchange identity (2.5). Prove that $\cdot$ and $\circ$ are equal, and that each is commutative.

Exercise 6: Combine Exercises 5 and 6 to prove that the fundamental group of a topological group is abelian.

2.6 Comma categories

Category of *objects under* b is the category $(b \downarrow C)$ with objects all pairs $\langle f, c\rangle$, where c is an object of C and $f \colon b \to c$ is and arrow of c. Arrows are arrows of C (resp. commutative triangles).

If a is an object of C, the category $(C \downarrow a)$ of *objects over* a has objects $f \colon c \to a$.

If b is an object of C and $S \colon D \to C$ a functor, the category $(b \downarrow S)$ of *objects S-under b* has an objects all pairs $\langle f, d\rangle$ with $f \colon b \to Sd$.

$(T \downarrow a)$ of objects T-over a.

Here is the general construction. Given categories and functors

$$E \xrightarrow{T} C \xleftarrow{S} D$$

the *comma category* $(T \downarrow S)$, also written (T, S) has objects all triples $\langle e, d, f\rangle$, with $d \in \mathrm{Obj}D$, $e \in \mathrm{Obj}E$, and $f \colon Te \to Sd$ and as arrows $\langle e, d, f\rangle \to \langle e', d', f'\rangle$ all pairs $\langle k, h\rangle$ of arrows $k \colon e \to e'$, $h \colon d \to d'$ such that $f' \circ Tk = Sh \circ f$. In pictures

$$\text{Objects } \langle e, d, f\rangle \begin{tikzcd} Te \arrow[d, "f"] \\ Sd \end{tikzcd} ; \qquad \text{arrows } \langle k, h\rangle \begin{tikzcd} Te \arrow[r, "Tk"] \arrow[d, "f"'] & Te' \arrow[d, "f'"] \\ Sd \arrow[r, "Sh"'] & Sd' \end{tikzcd} \tag{2.6}$$

with the square commutative. The composite $\langle k', h'\rangle \circ \langle k, h\rangle = \langle k' \circ k, h' \circ h\rangle$, when defined.

2.7 Graphs and Free Categories

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2.8 Quotient Categories

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3 Universals and limits

Definition 3.0.1. If $S\colon D \to C$ is a functor and c an object of C, a *universal arrow from c to S* is a pair $\langle r, u \rangle$ consisting of an object r od D and an arrow $u\colon c \to Sr$ of C, such that to every pair $\langle d, f \rangle$ with d an object of D and $f\colon c \to Sd$ an arrow of C, there is a unique arrow $f'\colon r \to d$ of D with $Sf' \circ u = f$. In other words, every arrow f to S factors uniquely through the universal arrow u, as in the commutative diagram

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[dr, "f"'] & Sr \arrow[d, dashed, "Sf'"] \\ & Sd \end{tikzcd} \tag{3.1}$$

Examples: Bases of vector spaces, free categories form graphs, fields of quotients, completion of metric space (universal for forgetful functor from complete metric spaces).

If D is a category and $H\colon D \to \mathbf{Set}$ functor, a *universal element* of the functor H is a pair $\langle r, e \rangle$ consisting of an object $r \in D$ and an element $e \in Hr$ such that for every pair $\langle d, x \rangle$ with $x \in Hd$ there is a unique arrow $f\colon r \to d$ of D with $(Hf)e = x$.

Diagonal functor: $\Delta\colon C \to C \times C$, $\Delta c = \langle c, c \rangle$.

3.1 The Yoneda Lemma

Proposition 3.1.1. *For a functor $S\colon D \to C$ a pair $\langle ru\colon c \to Sr \rangle$ is universal from form c to S if and only if the function sending each $f'\colon r \to d$ into $Sf' \circ u\colon c \to Sd$ is a bijection of hom-sets*

$$D(r, d) \cong C(c, Sd) \tag{3.2}$$

This bijection is natural in d. Conversely, given r and c, any natural isomorphism (3.2) is determined in this way by a unique arrow $u\colon c \to Sr$ such that $\langle r, u \rangle$ is universal from C to s.

Definition 3.1.2. Let D have small hom-sets. A *representation* of a functor $K\colon D \to \mathbf{Set}$ is a pair $\langle r, \psi \rangle$ with r an object of D and

$$\psi : D(r, -) \cong K \tag{3.3}$$

a natural isomorphism. The object r is called the *representing object.* The functor K is said to be *representable* when such a representation exists.

Proposition 3.1.3. *Let $*$ denote any one-point set and let D have small hom-sets. If $\langle r, u\colon * \to Kr \rangle$ is a universal arrow from $*$ to $k\colon D \to$ **Set**, then the function ψ which for each object d of D send the arrow $f'\colon r \to d$ to $K(f')(u^*) \in Kd$ is a representation of K. Every representation of K is obtained in this way from exactly one such universal arrow.*

The argument for Proposition 3.1.1 rested on the observation that each natural transformation $\varphi : D(r,-) \stackrel{\cdot}{\to} K$ is completely determined by the image under φ_r of the identity $1\colon r \to r$. This fact may be stated as follows:

Lemma 3.1.4 (Yoneda). *If $k\colon D \to \mathbf{Set}$ is a functor from D and r an object in D (for D a category with small hom-sets), there is a bijection*

$$y\colon Nat(D(r,-),K) \cong Kr \tag{3.4}$$

which sends each natural transformation $\alpha : D(r,-) \stackrel{\cdot}{\to} K$ to $\alpha_r 1_r$, the image of the identity $r \to r$.

Corollary 3.1.5. *For objects $r, s \in D$ each natural transformation $D(r,-) \stackrel{\cdot}{\to} D(s,-)$ has the form $D(h,-)$ for a unique arrow $h\colon s \to r$.*

Lemma 3.1.6. *The bijection of 3.4 is a natural isomorphism $y : N \stackrel{\cdot}{\to} E$ between the functors $E, N\colon \mathbf{Set}^D \times D \to \mathbf{Set}$.*

3.2 Coproducts and Colimits

Cokernels. Suppose that C has zero object z, so that for any two objects $b, c \in C$ there is a zero arrow $0\colon b \to z \to c$. The cokernel of $f\colon a \to b$ is then an arrow $u\colon b \to e$ such that (i) $uf = 0\colon ae$; (ii) if $h\colon b \to c$ has $hf = 0$, then $h = h'u$ for a unique arrow $h'\colon e \to c$. The picture is

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, "u"] \arrow[dr, "h"'] & e \arrow[d, dashed, "h'"] & uf = 0, \\ & & c & hf = 0. \end{tikzcd}$$

3.3 Products and limits

3.4 Categories with finite products

Proposition 3.4.1. *If a category C has a terminal object t and a product diagram $a \leftarrow a \times b \to b$ for any two of its objects, then C has all finite products. The product object provide, by $\langle a, b\rangle \to a \times b$, a bifunctor $C \times C \to C$. For any three objects there is an isomorphism*

$$\alpha = \alpha_{a,b,c}\colon a \times (b \times c) \cong (a \times b) \times c \tag{3.5}$$

natural in a, b and c. For any object a there are isomorphisms

$$\lambda = \lambda_a\colon t \times a \cong a \quad \varrho = \varrho_a\colon a \times t \cong a \tag{3.6}$$

which are natural in a, where t is the terminal object of C.

3.5 Groups in categories

Let C be a category with finite products products and a terminal object t. Then a *monoid* in C is a triple $\langle c, \mu\colon c \times c \to c, \eta\colon t \to c\rangle$, such that the following diagrams commute

$$\begin{tikzcd} c \times (c \times c) \arrow[r, "\alpha"] \arrow[d, "1\times\mu"'] & (c \times c) \times c \arrow[r, "\eta\times 1"] & c \times c \arrow[d, "\mu"] \\ c \times c \arrow[rr, "\mu"] & & c \end{tikzcd} \tag{3.7}$$

$$
\begin{tikzcd}
t \times c \arrow[r, "\eta\times 1"] \arrow[dr, "\lambda"'] & c \times c \arrow[d, "\mu"] & c \times t \arrow[l, "1\times\eta"'] \arrow[dl, "\varrho"] \\
& c &
\end{tikzcd}
\tag{3.8}
$$

μ=multiplication, α=asocitivity isomorphism of (3.5).

We now define a *group* in C to be a monoid $\langle c, \mu, \eta\rangle$ together with an arrow $\xi\colon c \to c$ which makes the diagram (with δ_c the diagonal)

$$
\begin{tikzcd}
c \arrow[r, "\delta_c"] \arrow[d] & c \times c \arrow[r, "1\times\eta"] & c \times c \arrow[d, "\mu"] \\
t \arrow[rr, "\eta"] & & c
\end{tikzcd}
\tag{3.9}
$$

commute (ξ=right inverse).

Proposition 3.5.1. *If C is a category with finite products, then an object c is a group (or, a monoid) in C if and only if the hom functor $C(-, c)$ is a qroup (respectively, a monoid) in the functor category* $\mathbf{Set}^{C^{op}}$.

4 Adjoints

4.1 Adjunctions

Definition 4.1.1. Let A and X be categories. An *adjunction* from X to A is a triple $\langle F, G, \varphi\rangle\colon XA$, where F and G are functors

$$
\begin{tikzcd}
F \arrow[r, shift left, leftarrow, "G"] \arrow[r, shift right, "F"'] & G
\end{tikzcd}
$$

while φ is a function which assigns to each pair of objects $x \in X$, $a \in A$ a bijection

$$\varphi = \varphi_{x,a}\colon A(Fx, a) \cong X(x, Ga) \tag{4.1}$$

which is natural in x and a.

An adjunction may also be described without hom-sets directly in terms of arrows. It is a bijection which assigns to each arrow $f\colon Fx \to a$ an arrow $\varphi f = \mathrm{rad} f\colon x \to Ga$, the *right adjunct of* f, in such a way that the naturality conditions

$$\varphi(f \circ Fh) = \varphi f \circ h, \quad \varphi(k \circ f) = Gk \circ \varphi f, \tag{4.2}$$

hold for all f and all arrows $h\colon x' \to x$ and $k\colon a \to a'$. It is equivalent to require that φ^{-1} be natural; i.e., that for every h, k and $g\colon x \to Ga$ one has

$$\varphi^{-1}(gh) = \varphi^{-1}g \circ Fh, \quad \varphi^{-1}(Gk \circ g) = k \circ \varphi^{-1}g. \tag{4.3}$$

Given such an adjunctions, the functor F is said to be a *left-adjoint* for G, while G is called a *right adjoint* for F. (TODO Find TEXequivalent!!!)

Every adjunction yields a universal arrow. Specifically, set $a = Fx$ in (4.1). The left hand hom-set of (4.1) then contains the identity $1\colon Fx \to Fx$; call its φ-image η_x. By Yoneda's

Proposition 3.1.1 this η_x is a universal arrow η_x for *every* objects x. Moreover, the function $x \mapsto \eta_x$ is a natural transformation $I_X \to FG$.

The bijection φ can be expressed in terms of the arrow η_x as

$$\varphi(f) = G(f)\eta_x \quad \text{for } f\colon Fx \to a. \tag{4.4}$$

Theorem 4.1.2. *An adjunction $\langle F, G, \varphi\rangle\colon X \to A$ determines*

(i) A natural transformation $\eta : I_x \stackrel{\cdot}{\to} GF$ such that for each object x the arrow η_x is universal to G from x, while the right adjunct of each $f\colon Fx \to a$ is

$$\varphi f = Gf \circ \eta_x \colon x \to Ga; \tag{4.5}$$

(ii) A natural transformation $\varepsilon\colon FG \to I_A$ such that arrow ε_a is universal to a from F, while each $g\colon x \to Ga$ has left adjunct

$$\varphi^{-1} g = \varepsilon_a \circ Fg \colon Fx \to a. \tag{4.6}$$

Moreover, both the following composites are the identities (of G, resp. F).

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G, \qquad F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F. \tag{4.7}$$

We call η *unit* and ε the *counit* of the adjunction.

Theorem 4.1.3. *Each adjunction $\langle F, G, \varphi\rangle\colon X \to A$ is completely determined by the items in any one of the following lists:*

(i) Functors f, G and a natural transformation $\eta : 1_x \stackrel{\cdot}{\to} GF$ such that each $\eta_x\colon x \to GFx$ is universal from G to x. The φ id defined by (4.5).

(ii) The functor $G\colon A \to X$ and for each $x \in X$ an object $F_0 x \in A$ and a universal arrow $\eta_x\colon x \to GF_0x$ from x to G. Then the functor F has object function F_0 and is defined on arrow $h\colon x \to x'$ by $GFh \circ \eta_x = \eta_{x'} \circ h$.

(iii) Functors F, G and a natural transformation $\varepsilon : FG \stackrel{\cdot}{\to} I_A$ such that each $\varepsilon_a\colon FGa \to a$ is universal from F to a. Here φ^{-1} is defined by (4.6).

(iv) The functor $F\colon X \to A$ and for each $a \in A$ an object $G_0 a \in X$ and an arrow $\varepsilon_a\colon FG_0a \to a$ universal from F to a.

(v) Functors F, G and natural transformations $\eta : I_X \stackrel{\cdot}{\to} GF$ and $\varepsilon : GF \stackrel{\cdot}{\to} I_A$ such that both composites (4.7) are the identity transformations. Here φ is defined by (4.5) and φ^{-1} by (4.6).

Corollary 4.1.4. *Any two left-adjoints F and F' of a functor $G\colon A \to X$ are naturally isomorphic.*

Corollary 4.1.5. *A functor $G\colon A \to X$ has a left adjoint if and only if, for each $x \in X$, the functor $X(x, Ga)$ is representable as a functor of $a \in A$. If $\varphi\colon A(F_0x, a) \cong X(x, Ga)$ is representation of this functor, then F_0 is the object function of a left-adjoint of G for which the bijection φ is natural in a and gives the adjunction.*

Theorem 4.1.6. *If the additive functor $G\colon A \to M$ between Ab-categories A and M has a left adjoint $F\colon M \to A$, then F is additive and the adjunction bijections*

$$\varphi\colon A(Fm, a) \cong M(m, Ga)$$

are isomorphisms of abelian groups (for all $m \in M$, $a \in A$).

4.2 Examples of Adjoints

4.3 Reflective Subcategories

Theorem 4.3.1. *For an adjunction $\langle F, G, \eta, \varepsilon\rangle\colon X \to A$:*

(i) G is faithful if and only if every component ε_a of the counit ε is epi,

(ii) G is full if and only if every ε_a is a split monic.

Hence G is full and faithful if and only if each ε_a is an isomorphism $FGa \cong a$.

Lemma 4.3.2. *Let $f^* : A(a,-) \dot{\to} A(b,-)$ be the natural transformation induced by an arrow $f\colon b \to a$ of A. Then f^* is monic if and only if f is epi, while f^* is epi if and only if f is a split monic (i.e., if and only if f has a left inverse).*

A subcategory A of B is called *reflective* in B when the inclusion functor $K\colon A \to B$ has a left adjoint $F\colon B \to A$. This functor F may be called a *reflector* and the adjunction $\langle F, K, \varphi\rangle = \langle F, \varphi\rangle\colon B \to A$ a *reflection* of B in its subcategory A.

4.4 Equivalence of Categories

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4.5 Adjoints for Preorders

skipped

4.6 Cartesian Closed Categories

To assert that a category C has all finite products and coproducts is to assert that the functors $C \to \mathbf{1}$ and $\Delta\colon C \to C \times C$ have both left and right adjoints. Indeed, the left adjoints give initial object and coproduct respectively, while the right adjoints give terminal object and product, respectively.

A category C with all finite products specifically given is called *cartesian closed* when each of the following functors

$$\begin{array}{ccc} C \to \mathbf{1}, & C \to C \times C, & C \overset{-\times b}{C}, \\ c \mapsto 0, & c \mapsto \langle c, c\rangle & , a \mapsto a \times b \end{array}$$

has a *specified* right adjoint (with a specified adjunction). This adjoints are written as follows

$$t \longleftarrow 0, \qquad a \times b \longleftarrow \langle a, b\rangle, \qquad c^b \longleftarrow c$$

TODO oprav to na otocena mapsto

The third required adjoint specifies for each functor $- \times b\colon C \to C$ a right adjoint, with the corresponding bijection

$$hom(a \times b, c) \cong hom(a, c^b)$$

natural in a and c. By the parameter theorem (to be proved in the next section), $\langle b, c\rangle \mapsto c^b$ is then (the object function of) a bifunctor $C^{op} \times C \to C$. Specifying the adjunction amounts to specifying for each c and b an arrow e

$$e\colon c^b \times b \to c$$

which is natural in c and universal form $- \times b$ to c. We call $e = e_b$ the *evaluation* map.

Exercise 1: a) If U is any set, show that the preorder $\mathcal{P}(U)$ of all subsets of U is a cartesian closed category.

b) Show that any Boolean algebra, regarded as a preorder, is cartesian closed.

Exercise 3: In any cartesian closed category, prove $c^t \cong c$ and $c^{b \times b'} \cong (c^b)^{b'}$.

Exercise 4: In any cartesian closed category obtain a natural transformation $c^b \times b^a \longrightarrow c^a$ which agrees in **Set** with composition of functions. Prove it (like composition) associative.

Exercise 5: Show that A cartesian closed need not imply A^J cartesian closed.