Fibonacciho a Lucasove čísla

Použité zdroje: [J, KM, Š]

$$F_n + F_{n+1} = F_{n+2} \tag{1}$$
$$F_0 = 0, \qquad F_1 = 1 \tag{2}$$

Podobne $L_n + L_{n+1} = L_{n+2}$ a $L_0 = 2$, $L_1 = 1$.
$F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \ldots$
Môžeme ich rozšíriť aj na záporné čísla: $F_{-1} = 1, F_{-2} = -1, F_{-3} = 2$.

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
F_n	-21	13	-8	5	-3	2	-1	1	$\boxed{0}$	1	1	2	3	5	8	13	21
L_n	47	-29	18	-11	7	-4	3	-1	$\boxed{2}$	1	3	4	7	11	18	29	47

Maticové vyjadrenie

Označme $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Potom

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} \tag{3}$$
$$A^n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \tag{4}$$

Pre Lucasove čísla môžeme dostať podobný vzťah

$$\begin{pmatrix} L_{n-1} & L_n \\ L_n & L_{n+1} \end{pmatrix} = A^n \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = A^n(2A - I) = 2A^{n+1} - A^n$$

z čoho vidíme

$$L_n = 2F_{n+1} - F_n = F_{n+1} + F_{n-1} \tag{5}$$

Všimnime si tiež, že $\chi_A(x) = x^2 - x - 1$, teda platí $A^2 - A - I = 0$.

Identity

Binetov vzorec:

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right), \tag{6}$$

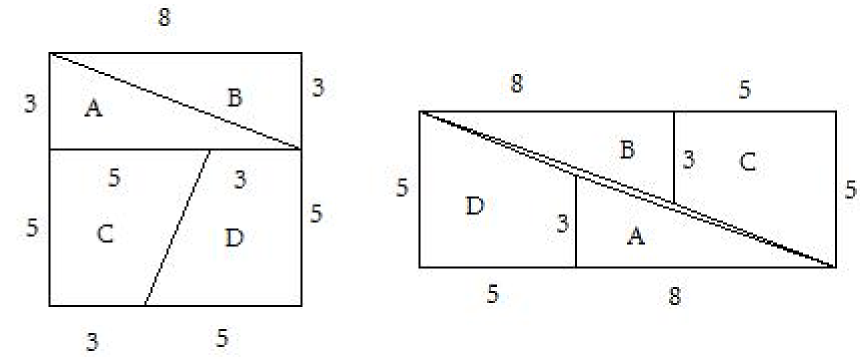
kde

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

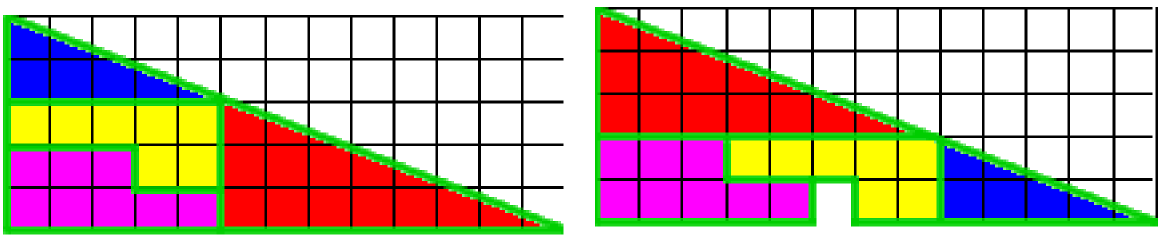
sú korene kvadratickej rovnice $x^2 - x - 1 = 0$, teda pre na platí

$$\lambda_1 + \lambda_2 = 1, \tag{7}$$
$$\lambda_1 \lambda_2 = -1. \tag{8}$$

$$L_n = \lambda_1^n + \lambda_2^n$$



Obr. 1: Geometrická interpretácia Cassiniho identity.



Obr. 2: Missing square puzzle

Cassiniho identita

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{9}$$

Príbuzná identita

$$F_{n-1}F_{n+2} - F_nF_{n+1} = (-1)^n$$

Konvolučná vlastnosť:

$$F_{m+n} = F_mF_{n+1} + F_{m-1}F_n \tag{10}$$

Z konvolučnej vlastnosti dostávame rovnosti:

$$F_{2n} = F_n(F_{n+1} + F_{n-1}) = F_nL_n = F_n(2F_{n+1} - F_n),$$
$$F_{2n+1} = F_{n+1}^2 + F_n^2,$$

ktoré umožňujú rýchly výpočet n-tého člena Fibonacciho postupnosti.

Pre ľubovoľné prirodzené čísla a, b, c, d také, že $a + b = c + d$ platí

$$F_aF_b - F_cF_d = (-1)^r(F_{a-r}F_{b-r} - F_{c-r}F_{d-r}) \tag{11}$$

Sumy s Fibonacciho číslami

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1 \tag{12}$$

Všeobecnejšie:

$$\sum_{i=0}^{n} F_i k^i = \frac{(k-1)k^{n+1}F_{n+1} - k^{n+2}F_{n+2} + k}{1-k-k^2}$$

$$\sum_{k=0}^{n} F_{2k+1} = F_{2(n+1)}$$
$$\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$$

$$\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$$
$$\sum_{k=0}^{n} F_k F_{k+1} = \frac{F_{n+1}^2 + F_n F_{n+1} + F_n^2 - 1}{2}$$

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k$$

Pre túto identitu nemám dôkaz založený na lineárnej algebre:

$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1} \tag{13}$$

Literatúra

[J] B. Johnson. Fibonacci numbers and matrices. `http://www.dur.ac.uk/bob.johnson/fibonacci/`.

[KM] Dan Kalman and Robert Mena. The Fibonacci numbers – exposed. *Math. Mag.*, 76(3):167–181, June 2003.

[Š] Beáta Štupáková. Fibonacciho a Lucasove čísla, 2008. bakalárska práca, FMFI UK, Bratislava.