JOURNAL OF COMBINATORIAL THEORY, Series B 26, 81–84 (1979)

Score Vectors of Tournaments

CHANG M. BANG AND HENRY SHARP, JR.

Department of Mathematics, Emory University, Atlanta, GA 30322

Communicated by the Editors

Received March 4, 1976

A tournament T on any set X is a dyadic relation such that for any $x, y \in X$ (a) $(x, x) \notin T$ and (b) if $x \neq y$ then $(x, y) \in T$ iff $(y, x) \notin T$. The score vector of T is the cardinal valued function defined by $R(x) = |\{y \in X : (x, y) \in T\}|$. We present theorems for infinite tournaments analogous to Landau's necessary and sufficient conditions that a vector be the score vector for some finite tournament. Included also is a new proof of Landau's theorem based on a simple application of the "marriage" theorem.

1. INTRODUCTION

Let X be a non-empty arbitrary set. It will be convenient to identify a tournament on X with its characteristic function in the form of a (0, 1)-matrix $T = (t_{xy}: x, y \in X)$ in which (a) $t_{xx} = 0$ for all x, and (b) $t_{xy} + t_{yx} = 1$ for all $x \neq y$. Whatever the cardinality of X, T is sometimes called an $|X|$-tournament, and is finite (infinite) if X is finite (infinite). For each $x \in X$, the x-th score of T is the cardinal $R(x) = r_x = |\{ y \in X: t_{xy} = 1\}|$, and the vector $R = (r_x: x \in X)$ is the score vector of T. We are concerned in this paper with necessary and sufficient conditions that a cardinal valued function R be the score vector of a tournament on some set X. In Section 2 we give a new proof of Landau's theorem [5] for finite tournaments, and in Section 3 we show that this technique generalizes to prove analogous theorems for $\aleph_0$-tournaments. We also extend our results to tournaments on uncountable sets.

2. FINITE TOURNAMENTS

Each component of the score vector of any finite tournament is a non-negative integer, but not every such integer-valued vector qualifies as a score vector. In 1953, Landau characterized score vectors of finite tournaments (Theorem 1, below). Other proofs of this theorem appear in the literature (Alway [1]; Brauer, Gentry, Shaw [2]; Ryser [6]), all depending upon rather complicated arguments. We give a very short proof below, based upon a

0095-8956/79/010081-04$02.00/0

simple application of Philip Hall's theorem on the existence of systems of distinct representatives [3].

THEOREM 1. (Landau [5]). *Let $X = \{1, 2,..., n\}$, and let $R = (r_1, r_2,..., r_n)$ be a vector of nonnegative integers. There exists a tournament on X having score vector R if, and only if,*

(a) *for each nonempty subset $Y \subseteq X$, $\sum_{k \in Y} r_k \geqslant \binom{|Y|}{2}$, and*

(b) $\sum_{k \leqslant n} r_k = \binom{n}{2}$.

Proof. Necessity is clear, and to prove sufficiency let R satisfy the given conditions. For each $k = 1, 2,..., n$, let G_k be a set with $|G_k| = r_k$. Assume that the sets G_k are pairwise disjoint and set $G = \bigcup_{k \leqslant n} G_k$.

Let $B = \{(i, j) | 1 \leqslant i < j \leqslant n\}$, and $H = \{G_i \cup G_j | (i, j) \in B\}$. Note that H is a family of subsets of G with indexing set B. For each nonempty subset A of B let $Y = \{k | (k, \cdot) \text{ or } (\cdot, k) \in A\}$. Each 2-element subset of Y corresponds to at most one element of A, hence

$$|A| \leqslant \binom{|Y|}{2} \leqslant \sum_{k \in Y} r_k = \Big|\bigcup_{k \in Y} G_k\Big| = \Big|\bigcup_{(i,j) \in A} (G_i \cup G_j)\Big|.$$

By applying Hall's theorem to the family H we obtain an injection $f: B \to G$ such that $f(i, j) \in G_i \cup G_j$.

We now construct the tournament $T = (t_{ij})$ as follows: let $t_{ii} = 0$ for all i, and for all $i < j$ let

$$\begin{cases} t_{ij} = 1 \quad \text{and} \quad t_{ji} = 0 \quad \text{if} \quad f(i, j) \in G_i, \\ t_{ij} = 0 \quad \text{and} \quad t_{ji} = 1 \quad \text{if} \quad f(i, j) \in G_j. \end{cases}$$

Finally, observe that $\sum_{l \leqslant n} t_{kl} \leqslant |G_k| = r_k$ for each k. But by condition (b) of the hypothesis, $|B| = |G|$, hence f is a bijection and thus $\sum_{l \leqslant n} t_{kl} = r_k$ for all k. This completes the proof. ∎

3. TOURNAMENTS ON INFINITE SETS

Assume first that X is countable, $X = \{1, 2, 3,...\}$, and note that the k-th score of a tournament T on X need not, in general, be finite. A tournament in which each score *is* finite will be called *row-finite*, and with respect to such tournaments we may apply the infinite version of Hall's theorem as the initial step in our construction.

THEOREM 2. *Let $X = \{1, 2, 3,...\}$, and let $R = (r_1, r_2, r_3,...)$ be a vector of nonnegative integers. Then, there exists a row-finite tournament T on X with score vector R if, and only if, for each nonempty finite subset $Y \subseteq X$, $\sum_{k \in Y} r_k \geqslant \binom{|Y|}{2}$.*

Proof. Necessity of the condition is obvious since the submatrix on any nonempty subset $Y \subseteq X$ is itself a tournament. For the proof of sufficiency, assume R satisfies the condition. Then exactly as in the proof of Theorem 1, we define the sets G_k, G, B, and H. Now H is an infinite family of finite sets indexed over B to which the infinite case of Hall's theorem [4] applies, hence there exists an injection $f: B \to G$ such that $f(i,j) \in G_i \cup G_j$ for all $i < j$. Again we construct a tournament $T = (t_{ij})$ in the same way, and note that if $S = (s_1, s_2, s_3, ...)$ is the score vector for T, then $s_k \leqslant r_k$ for all k. Contrary to the previous case, however, we cannot conclude that f is a bijection. If $s_k < r_k$ for any set of subscripts, then the tournament guaranteed by Hall's theorem will be modified inductively to produce a tournament having the desired score vector.

Let p be the least integer such that $s_p < r_p$. Replace by 1's the first $r_p - s_p$ 0's in the p-th row of T having column subscripts $> p$. This increases the p-th score precisely to r_p. Replacing by 0's the corresponding $r_p - s_p$ 1's in the p-th column results in a tournament T' with a score vector $S' = (s_1', s_2', s_3', ...)$ satisfying

$$s_i' = r_i \qquad \text{if} \quad i \leqslant p,$$

$$s_i' \leqslant r_i \qquad \text{if} \quad i > p.$$

Continue this process and obtain a tournament with the prescribed score vector R. ∎

If T is a countable tournament which is not row-finite, then a simple modification of the above statement is necessary. We omit the proof, which follows readily upon partitioning X into F and $X \backslash F$, and then using, if necessary, the inductive procedure described above.

THEOREM 3. *Let $X = \{1, 2, 3, ...\}$, let $R = (r_1, r_2, r_3, ...)$ be a vector each component of which is either $\aleph_0$ or a non-negative integer, and let $F = \{k: r_k < \aleph_0\}$. There exists a tournament T on X with score vector R if, and only if, for each nonempty finite subset $Y \subset F$,*

$$\sum_{k \in Y} r_k \geqslant \binom{|Y|}{2}.$$

It is an interesting observation that there cannot exist a row-finite tournament on an uncountable set: the Landau condition implies (roughly) that in order for R to be a score vector, it must not contain too many small components. In particular, for finite X the vector R must not contain more than $2m + 1$ components each $\leqslant m$, and for infinite X the constraint is specified in the following easily proved lemma (α^+ denotes the successor to the cardinal α).

LEMMA 4. *If T is a tournament on the infinite set X (of arbitrary cardinality), then for each infinite cardinal $\alpha < |X|$,*

$$|\{x \in X: r_x \leqslant \alpha\}| \leqslant \alpha^+.$$

Now if X is an infinite set and $R = (r_x: x \in X)$ is a vector of *infinite* cardinals with $r_x \leqslant |X|$, then the authors have shown by transfinite induction that it is possible to construct a tournament T on X having score vector R provided R satisfies both the condition in Lemma 4 and

$$|\{x \in X: r_x \geqslant \alpha\}| = |X|.$$

Combining Theorem 3 with this result yields the following general theorem.

THEOREM 5. *Let X be an uncountable set, $R = (r_x: x \in X)$ be a vector of cardinals, and $F = \{x \in X: r_x < \aleph_0\}$. Then there exists a tournament on X with score vector R if, and only if,*

(a) $\sum_{y \in Y} r_y \geqslant \binom{|Y|}{2}$ *for each finite set* $Y \subseteq F$,

(b) $|\{x \in X: r_x \leqslant \alpha\}| \leqslant \alpha^+$ *for each infinite cardinal* $\alpha \leqslant |X|$, *and*

(c) $|\{x \in X: r_x \geqslant \alpha\}| = |X|$ *for each cardinal* $\alpha < |X|$.

REFERENCES

1. G. G. ALWAY, Matrices and sequences, *Math. Gaz.* **46** (1962), 208–213.
2. A. BRAUER, I. C. GENTRY, AND K. SHAW, A new proof of a theorem by H. G. Landau on tournament matrices, *J. Combinatorial Theory* **5** (1968), 289–292.
3. P. HALL, On representatives of subsets, *J. London Math. Soc.* **10** (1935), 26–30.
4. P. R. HALMOS AND H. E. VAUGHAN, The marriage problem, *Amer. J. Math.* **72** (1950), 214–215.
5. H. G. LANDAU, On dominance relations and the structure of animal societies, III. The conditions for a score structure, *Bull. Math. Biophys.* **15** (1953), 143–148.
6. H. J. RYSER, Matrices of zeros and ones in Combinatorial Mathematics, *in* "Recent Advances in Matrix Theory" (H. Schneider, Ed.), pp. 103–124.) Publ. No. 12, Mathematics Research Center, United States Army, The University of Wisconsin, 1964.