Hilbert's Program and Gödel's Incompleteness Theorems

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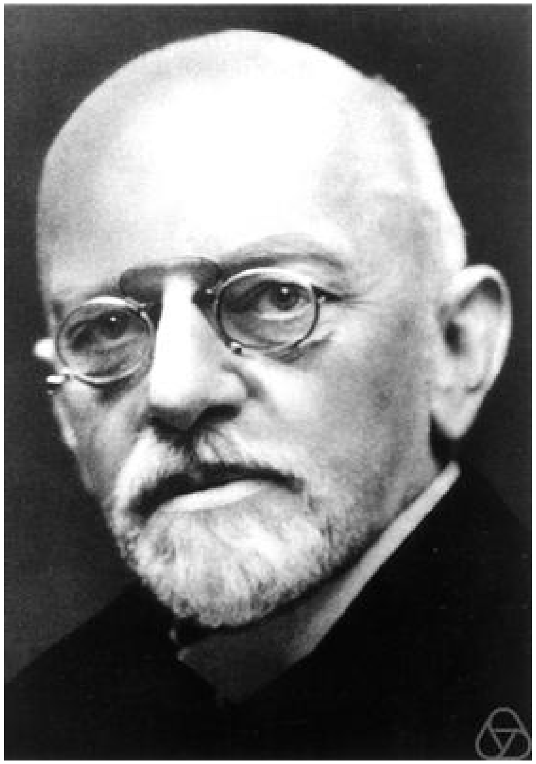
Main characters

Kurt Gödel
(1906 – 1978)



David Hilbert
(1862 – 1943)

“Mythical” interpretations of Gödel’s Incompleteness Theorems

- Human knowledge is inevitably incomplete.
- Human knowledge cannot be enclosed within any formal system given in advance.
- Human intellect, due to its capability of reflection, is superior to whatever computer.
- It cannot be guaranteed in advance that our accounts will not lead to a logical contradiction.
- We never can be absolutely sure that mathematics is free of contradictions.

Bare fact

Gödel's Incompleteness Theorems

prove the impossibility to carry out

Hilbert's Program in its original form.

The popular myth widely spread (even among mathematicians)

- Hilbert's Program of formalization of mathematics should have transformed mathematics into a mere game with symbols according to certain formal rules, deprived of any meaning, and that way to eliminate the need of creative human activity from it.
- Thanks to Gödel we can state with satisfaction that this naïve, dryasdust and necrophilic Hilbert's aim failed, so that we, *creative* mathematicians, are and for ever will be indispensable for the progress of human thought.

What will follow?

- Circumstances of and the reasons for the rise of Hilbert's Program
- Goals of Hilbert's Program
- Formulation of Gödel's Theorems and basic ideas of their proof
- The relation of Gödel's Theorems and Hilbert's Program

The unity of mathematics

In modern period the inner unity of mathematics is strongly perceived as a consequence of several discoveries:

- Descart's discovery of the co-ordinate system, unifying algebra and analysis on one hand and geometry on the other (1637).
- The discovery and elaboration of infinitesimal calculus, enabling to grasp the motion and formulate physical laws by mathematical means, as well as to master complex geometrical formations by elementary ones (Newton, Leibniz, Euler, Laplace, Lagrange, Hamilton, Gauss, Cauchy,... — approx. 1670–1830).
  - ⋆ *However, the use of infinitesimal and infinitely large magnitudes frequently causes difficulties — the attempts to formulate rigorous rules* "what is allowed and what is not" *failed.*
- Riemann's (1854) and Klein's (1872) unifying view upon various types of geometries (Euclid, Gauss, Bolyai, Lobachevski, Cayley,...).

There grows the need to guarantee this manifested and perceived unity of mathematics an "institutionalized" form.

Reasons for the acceptance of Cantor's set theory and reconstruction of mathematics within the universe of sets at the turn of the 19th and 20th century

- Set theory provides the entire mathematics with a universal language and, in the universe of sets, it grants all mathematical branches a generous space for presenting, modeling and/or set-theoretical reconstruction of all domains of objects studied by them up to that time.
  - ⋆ According to Hilbert: *A paradise created for us by Cantor.*
- Troubles and paradoxes of the original infinitesimal calculus, caused by the use of infinitely small and infinitely big magnitudes, definitively(?) overcome by Weierstrass (approx. 1850).
  - ⋆ Mathematical analysis based on the notion of limit ($\varepsilon\delta$-analysis).
- The set-theoretical models of real numbers, constructed by Cantor and Dedekind (approx. 1870–1875) grant a solid ground for the $\varepsilon\delta$-analysis.

What's the price?

- For the exclusion of infinitesimals and infinitely big quantities one has to pay by the acceptance of the *actually infinite* sets.
  - ⋆ Even individual real numbers are modeled as actually infinite sets of rational numbers.
- The prevalent conception of infinity up to that time was namely the *potential infinity*.
- Formulation of analysis in the language of limits and $\varepsilon\delta$ seemingly returns to the original, potential conception:
  - ⋆ "$(\forall\,\varepsilon)(\exists\,\delta)(\ldots)$" evokes the illusion of a process.

Crisis of set theory and the foundations of mathematics caused by the discovery of paradoxes

- Some classical paradoxes, known already from antique times (as, e.g., the *Liar's Paradox*), as well as some recent paradoxes can be reproduced both within Cantor's set theory and within Frege's logical system of arithmetic (*Begriffsschrift*).
- Here, however, they turn into logical contradictions.
  - ⋆ Their discovery in the end of the 19th century causes a shocking effect.
- The "principal offender" is Cantor's naïve *scheme of comprehension*:
  - ⋆ For any property $P(x)$ one can form the set $\{x \mid P(x)\}$.
- For instance, for Russel's set $R = \{x \mid x \notin x\}$ we have

$$R \in R \;\Leftrightarrow\; R \notin R \quad \text{— a contradiction.}$$

  - ⋆ This is the set-theoretical version of *Liar's Paradox*:

"This sentence is not true."

- On one hand, there arises a belief that the paradoxes can be avoided and overcome, on the other hand, there are concerns that the actually infinite sets *inevitably* cause analogous problems as once did the infinitesimal and infinitely big quantities.

Proposed ways out

- *Logicism*, i.e., the reconstruction of mathematics as a part of logic.
- Russell and Whitehead in *Principia Mathematica* (1910–13), building on Frege's ideas:
  - ⋆ Restriction of the comprehension scheme within the *theory of types*.
  - ⋆ Latter on, Quine in *New Foundations*, for *stratified formulas* $P(x)$, only.
- *Zermelo's axiomatic system* of Cantor's set theory (1908), later on extended by Fraenkel (ZF and ZFC respectively, i.e., ZF + Axiom of Choice)
  - ⋆ restriction of the comprehension scheme just to $\{x \in A \mid P(x)\}$ for any *set-theoretical formula* $P(x)$ and a set A given in advance (three exceptions: *axioms of pair, union and power set*).

Negative reactions

- Rejection of actual infinity (Poincaré).
- In radical form *intuitionism* (Brouwer 1908-10).
- Casting doubt upon some laws of classical logic:
  - $\star$ the law of the excluded middle $A \vee \neg A$;
  - $\star$ the law of the double negation $\neg\neg A \Rightarrow A$;
  - $\star$ the quantifier rule $\neg(\forall\, x)\neg P(x) \Rightarrow (\exists\, x)P(x)$.
- Rejection of the axiom of choice and of non-constructive existential proofs in general.
- Danger of a drastic reduction of classical mathematics.

A bit of philosophy

How is it like indeed? Do there exist actually infinite sets or not?

- Hilbert admits that the actual infinity is not realized by any grouping of objects in the real world.

However, that's not the point!

- Owing mainly to the *quantum mechanics*, Hilbert is aware that even in physics by far not all theoretical constructions have to directly correspond to real objects or phenomena.
- In mathematics this has been clear for long: e.g., zero and negative integers, imaginary numbers, improper points and the improper line in projective geometry, etc.
  - ⋆ Intuitive insights usually arise later on (if at all).
- Thus from the mathematical point of view that question makes no sense and the controversies it gave rise to are idle!
- The philosophical thesis behind this conception: In mathematics *existence* = *consistency* (i.e., the absence of logical contradiction).

Correct questions

- Is the abstract idea of the actual mathematical infinity useful?

  Yes, for sure!

- Is it possible to grasp the infinite sets within a consistent system of thought?

  That's problematic, nevertheless, we believe that it is possible.

  However, namely this requires a proof!

Facing the challenge: Hilbert's Program

- Hilbert's Program is a serious attempt to find for mathematics a way out from the crisis caused by the discovery of paradoxes and, at the same time, a defense of set theory, classical logic and mathematics based on them against the attack from the intuitionistic position.
- The goal is to elaborate such an axiomatic system of set theory which would make possible to incorporate the "whole" of mathematics, and to prove its *consistency* and *completeness* by strictly *finitistic* methods, i.e., by means which even the intuitionists would have to accept.
- There's a hope that the Zermelo-Fraenkel axiomatic system ZF, maybe extended by the axiom of choice and, perhaps, by some additional properly designed axioms, already satisfies these requirements.

Consistency

- Consistency of a first order theory T means:

  It cannot happen that for some sentence (formula without free variables) A in the language of T both A and its negation $\neg A$ were provable in T.

- The contradictions based on the known paradoxes can be reproduced neither in *Principia Mathematica* by Russell and Whitehead nor in the Zermelo-Frankel axiomatic system.

- Unfortunately, that's not enough:

  $\star$ It is necessary to present a *proof of consistency*, i.e., to prove that in the relevant axiomatic system no contradictions can occur.

How all this can be done?

- For some theories it suffices to present a finite model.
- However, for theories with aspiration to play the role of the *foundations of mathematics* this possibility is out of the question.
- Therefore, we have to look more closely at the nature of *mathematical proofs*.
- The objects of classical mathematics are numbers, functions, geometrical formations, algebraic expressions, vector fields, etc., as well as their properties and relations between them — but not the *proofs* themselves.
- *Proofs* are expressions of processes of thought by means of which we establish some statements, properties or relations as logically necessary consequences of some other statements, properties or relations (as a rule more elementary or more evident ones).

Formalization

- Mathematical theories can be rewritten (some in a more, some in a less natural way) in a symbolic language (Frege).
- *Formulas*, i.e., the written expressions of statements, properties and relations, are "just" finite strings of symbols formed according to some combinatorial rules.
- *Formal proofs* can be written as finite sequences of formulas fulfilling some mechanically verifiable conditions.
- *Proof theory* or *metamathematics* is already a *branch of mathematics* dealing with *formal* proofs.
- *Consistency* of a formal theory T now means:
  In T there do not exist two proofs (i.e., certain finite sequences of formulas) one of which terminates by some sentence A and the other by its negation $\neg A$.

To prove the consistency of some formal theory T finally becomes a well formulated mathematical problem.

Completeness

- Completeness of a first order theory T means:

  For any sentence A in the language of T exactly one of the statements A, $\neg A$ is provable in T.

- To prove the completeness of a formal theory T is once again a well defined metamathematical task.

- In mathematics formulated within some *complete* axiomatic system of its foundations the *law of the excluded middle* would hold true even in the intuitionistic sense.

The finitistic requirement...

- The intuitionists must be defeated by their own arms!
- That's why the proofs of consistency and completeness of the theory forming the foundations of mathematics (as well as all the means employed in metamathematics) have to comply with the following *finitistic requirement*:
  - ⋆ They must not make use of infinite sets, the law of excluded middle, the axiom of choice, non-effective existential proofs, etc., i.e., the means which they first have to justify *before using them.*
- The finitistic position (Hilbert, Herbrand) makes the intuitionistic requirement of constructiveness more precise and even *more strict*.

... and its consequences

- A finitistic and constructive proof of completeness requires in fact the elaboration of a universal decision procedure (algorithm), hence it has to include the proof of *decidability* of the relevant axiomatic theory, which should play the role of the foundations of mathematics.
- Hilbert himself is firmly convinced in the solvability (decidability) of any sufficiently clearly formulated mathematical problem.
- An additional requirement is the proof of *conservativeness* of non-effective infinitistic methods:

  Everything what can be proved using them, can be proved without them, as well, just for the cost of longer and more complicated proofs.
- That way the reason for forbidding them would be removed:

  That would now appear as a stubborn reactionary tendency, refusing on irrational grounds the use of new, more efficient tools of set-theoretical mathematics.

Gödel's Completeness Theorem (1929–30)

- Gödel works within the guidelines of Hilbert's Program.
- In his dissertation he proves the *Completeness Theorem* for the first order logic (predicate calculus):

  Every consistent first order theory has some model.
- Logical (necessary) consequences of axioms of a first order theory are exactly those statements in its language, which can be derived from these axioms by formal proofs.
- Closing the process of reflection of mathematical thought in mathematical logic as a branch of mathematics.
- Bonus — *Compactness Theorem*:

  If every finite family of axioms of a theory T has a model, then T as a whole also has a model.

  ⋆ Strong tool enabling to construct models of first order theories, frequently "out of nothing".
- Minor flaw:

  ⋆ The proof of the Completeness Theorem does not (and cannot) comply with the finitistic requirement.

Technical preconditions of Gödel's Incompleteness Theorems

- Transition from *formalizaton* to *arithmetization* of provability.
- Assignment of *numerical codes* to symbols and expressions (i.e., finite strings of symbols) of certain formal language.
- Description of properties of expressions
  - ⋆ *"to be a formula"*,
  - ⋆ *"to be an axiom of given theory"*,
  - ⋆ *"to be a proof within that theory"*,

  etc., in terms of certain arithmetical properties of their codes.
- Fully in the spirit of Leibniz's project
  - ⋆ *characteristica universalis*,
  - ⋆ *calculus ratiocinator*,
  - ⋆ *ars iudicandi*.

To what do Gödel's Incompleteness Theorems apply?

To any first order theory T, such that

- T is *recursively axiomatizable*, i.e., its axioms form a finite list or at least can be effectively recognized;
- *Peano arithmetics* (PA), i.e., the basic theory of addition and multiplication of natural numbers, can be interpreted in T.

Axioms of PA

$$0+1=1, \qquad 0 \neq x+1, \qquad x+1=y+1 \;\Rightarrow\; x=y;$$
$$x+0=x, \qquad x+(y+1)=(x+y)+1;$$
$$x \cdot 0 = 0, \qquad x \cdot (y+1) = (x \cdot y) + x.$$

Scheme of mathematical induction:

$$\Big[F(0) \wedge (\forall\, x)\big(F(x) \Rightarrow F(x+1)\big)\Big] \;\Rightarrow\; (\forall\, x) F(x).$$

where $F(x)$ is an arbitrary formula in the language of PA.

Using Gödel's coding it is possible...

- ... to line up formulas with at most one free variable x, expressing properties of natural numbers, into a sequence $F_0(x),\ F_1(x), F_2(x), \ldots, F_n(x), \ldots;$
- ... to line up proofs in the theory T into a sequence $\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_k, \ldots;$
- ... to define in T an arithmetical and effectively decidable relation $P(x, y, z)$ such that for any natural numbers k, m, n the following condition is satisfied:

Δ_k is a proof of the statement $F_n(m)$ in the theory T

if and only if $P(m, n, k)$ holds.

Gödel's shift of Liar's Paradox

- Let g be the number of the formula $\neg(\exists\, z)P(x, x, z)$;
  then $F_g(g)$ is the formula $\neg(\exists\, z)P(g, g, z)$.
- The sentence $F_g(g)$ is "just" an arithmetical statement about natural numbers, similarly as, e.g.:

  "The equation $x^3+y^3+z^3 = 33$ *has no solution in positive integers."*

- On the other hand, it codes the statement:

  "There does not exist any proof of the sentence $F_g(g)$*."*

- That way Gödel's statement $F_g(g)$ can be identified with the self-referential statement:

  "This statement is unprovable in T*."*

  Or:

  "I am unprovable in T*."*

Gödel's statement is necessarily true...

... more precisely, it is the case if T is *arithmetically correct*, i.e., all statements about natural numbers provable in T are true.

- Namely, if Gödel's statement $F_g(g)$ were not true, then it would be provable in T.
- That way $F_g(g)$ would be an arithmetical statement provable in T.
- However, all arithmetical statements provable in T are true.
- That contradicts our starting assumption that Gödel's statement is not true.
- Hence Gödel's statement $F_g(g)$ is true but unprovable in T.

Preliminary version of Gödel's First Incompleteness Theorem

In every recursively axiomatizable arithmetically correct first order theory T there are true *statements about natural numbers which are not provable in T.* (Namely Gödel's statement $F_g(g)$ is such.)

Objection:

- We just have *proved* that Gödel's statement is true.
- Hence it is provable and, at the same time, what it states is true, thus it is unprovable.
- That way we once again obtained a contradiction, very similar to Russell's version of *Liar's Paradox*.

Response:

- This time it is not a contradiction *within the formal system* of the theory T.
- Our "proof" of the truth of Gödel's statement is namely not a proof in T but just an informal argumentation showing that Gödel's statement is intuitively true.

Gödel's First Incompleteness Theorem

Let T be any recursively axiomatizable first order theory such that PA is interpretable in T.

- *If T is consistent then the statement $F_g(g)$ is not provable in T.*
- *If T is additionally ω-consistent, then neither the statement $\neg F_g(g)$ is provable in T.*

In that case T is incomplete.

- *ω-consistency* is a technical condition, stronger than mere consistency.
- Rosser (1936) proved that we can manage without this assumption.
  - ⋆ He constructed an example of a sentence B, such that already from the assumption of consistency of the theory T it follows that both B and $\neg B$ are unprovable in T.
  - ⋆ Rosser's sentence B codes the self-referential statement:

    "If I am provable in T, then among the proofs, preceding my proof in the sequence $\Delta_0, \Delta_1, \Delta_2, \ldots$, there is a proof of my negation."

Unprovability of consistency

- If s is the number of the formula $x \neq x$, then $F_s(0)$ is the statement $0 \neq 0$.
- Obviously, T is consistent if and only if the statement $F_s(0)$, i.e., $0 \neq 0$, is not provable in T.
- That way the arithmetical sentence $\neg(\exists\, z)P(0, s, z)$, which we abbreviate as $\mathrm{Cons}(T)$, codes the statement of consistency of the theory T.
- The implication $\mathrm{Cons}(T) \Rightarrow F_g(g)$ is provable in T.
- Hence, if the sentence $\mathrm{Cons}(T)$ were provable in T, then so would be Gödel's statement $F_g(g)$, contradicting Gödel's First Incompleteness Theorem.

Gödel's Second Incompleteness Theorem

Let T be any recursively axiomatizable first order theory such that PA is interpretable in T. If T is consistent, then the sentence $\mathrm{Cons}(T)$*, coding the statement of consistency of T, is not provable in T.*

- That way the sentence $\mathrm{Cons}(T)$ (together with Gödel's statement $F_g(g)$) is another example of a true arithmetical proposition unprovable in T.
- In ZF, but even in considerably weaker theories, it is possible to prove the consistency of PA:
  - ⋆ Gentzen's proof of consistency of PA using *transfinite induction* over ordinals $< \varepsilon_0 = \omega^{\omega^{\omega^{\cdots}}}$ (1936);
  - ⋆ Gödel's proof using *recursive functionals* (1958).
- Infinitary methods of set theory, but even some much weaker ones, are *not* conservative extensions of finitistic methods (the latter ones can be formalized within PA).

Next following results

Tarski's Theorem on Undefinability of Truth

(more precisely, of the *satisfaction relation*)

There does not exist any formula $S(x, y)$ *in the language of PA, such that, for any natural numbers* m*,* n*, the statement* $F_n(m)$ *is true if and only if* $S(m, n)$ *holds.*

Church-Turing Undecidability Theorem

Peano arithmetic PA, as well as the predicate calculus itself, are algorithmically undecidable.

- The same is true for any consistent recursively axiomatizable theory T in which PA can be be interpreted.

Failure or vindication of Hilbert's Program?

The above results establish the impossibility to carry out Hilbert's Program in every one of its aims.

- However, it would be a mistake to conclude that Hilbert's Program was just a naïve dreaming to which Gödel delivered the deserved lesson.
- Just the opposite, Hilbert formulated fundamental questions, and in that time he had good reasons to expect the answers he anticipated.
- Without his clear-cut formulated questions we most probably would not learn the surprising responses so soon.
- His program gave rise to *Metamathematics* or *Proof Theory* as a new independent and fruitful branch of mathematics, which marked significantly the development of some branches of theoretical computer science (*theory of formal languages, recursive functions and computability theory, program analysis, automatical theorem proving*...).

Is Arithmetic or Set Theory (in)consistent?

- Gödel's Second Incompleteness Theorem does not state, that it is impossible to prove the consistency of Peano Arithmetic — it is just impossible to prove it within PA itself.
  - ⋆ Recall the proofs of consistency of PA by Gentzen (1936) and by Gödel himself (1958).
- And the more, Gödel's Theorems do not imply that arithmetic, or even mathematics as a whole, were inconsistent.
- On the other hand, the possibility of a contradiction in some axiomatic system of set theory, like ZFC, NBG (von Neumann-Gödel-Bernays), KM (Kelly-Morse), extended by, say, certain strong axioms of existence of *big cardinals* cannot be completely excluded *a priori*.
- However, the attempts to infer from Gödel's Theorems some too far reaching conclusions of extra-mathematical character purport even more problematically.