Pontryagin-van Kampen Duality and Fourier Transform in Hyperfinite Ambience Gordon's Conjectures

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- G is totally disconnected iff each $\gamma \in \widehat{G}$ generates a relatively compact subgroup

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  $(f * g)(x) = \int f(x - t)\, g(t)\, \mathrm{d}\boldsymbol{m}_G(t)$
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$\boldsymbol{m}_G$, $\boldsymbol{m}_{\widehat{G}}$ denote *Haar measures* on G, $\widehat{G}$, resp.

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  for $f \in \mathrm{L}^1(G) = \mathrm{L}^1(G, \boldsymbol{m}_G)$, $\gamma \in \widehat{G}$
- $\widehat{f}$ is the **Fourier transform** of f, $\quad \big\|\widehat{f}\,\big\|_\infty \le \|f\|_1$
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- proper normalization of Haar measures $\boldsymbol{m}_G$, $\boldsymbol{m}_{\widehat{G}}$ ensures **Plancherel identity** and **Fourier inversion formula**
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Isn't there some "universal extension" of all the spaces $\mathrm{L}^p(G)$ $(1 \leq p \leq 2)$ and $\mathrm{M}(G)$, and a uniform scheme defining the Fourier transform on this extension, covering all the particular cases, like if G were finite?

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P. Vopěnka: *Odmítnutí Newtonova a Lebnizova pojetí infinitesimálního kalkulu matematiky 19. a 20. století – vyvolané ať již jejich neochotou či neschopností domyslet a dotvořit základní pojmy, o než se původní pojetí tohoto kalkulu opíralo – bylo jedním z největších omylů nejen matematiky, ale evropské vědy vůbec.*

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- every consistent “not too big” system of standard formulas is satisfied by some object (*saturation*)

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Nonstandard Analysis 3

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D. Zeilberger: *Continuous analysis and geometry are just degenerate approximations to the discrete world [...] While discrete analysis is conceptually simpler (and truer) than continuous analysis, technically it is (usually) much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics [...]*

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- X_{f} and E are external sets; X_{f} is union and E is intersection of "not too many" internal sets (Σ^0_1 and Π^0_1)

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- for $\mathbf{X}$ locally compact, compact subsets of $\mathbf{X}$ are exactly $A^\flat = \{{}^\circ a;\ a \in A\}$ for internal $A \subseteq X_{\mathrm{f}}$ ("pushing-down" A)

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- $\mathbf{f} \in \mathrm{C_0}(\mathbf{X})$ iff $\mathbf{f}$ has lifting $f \in \mathcal{C}_0(X, E, X_{\mathrm{f}})$, i.e., $f \in \mathcal{C}_{\mathrm{b}}(X, E, X_{\mathrm{f}})$ and $f(x) \approx 0$ for $x \in X \setminus X_{\mathrm{f}}$

Nonstandard Analysis 9

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Some of them represent standard objects of different nature: cosets of functions in Lebesgue L^p spaces, measures, distributions, etc.

Nonstandard Analysis 10

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- $A \mapsto {}^\circ\nu_d(A)$, where ${}^\circ\nu_d(A) = \infty$ if $\nu_d(A) \notin \mathbb{F}^*\mathbb{R}$, induces the **Loeb measure** λ_d on X, defined on (the completion of) the σ-algebra generated by internal subsets of X (*a la* Caratheodory)

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Loeb measure.

X is hyperfinite set, $d : X \to {}^*\mathbb{R}$ is internal, $d(x) \geq 0$ for $x \in X$; typically, $d(x) = d$ is uniform (constant) on X

- d defines (hyper)discrete weighted counting measure $\nu_d(A) = \sum_{a \in A} d(a)$ on the Boolean algebra ${}^*\mathcal{P}(X)$ of all internal subsets of X
- $A \mapsto {}^\circ\nu_d(A)$, where ${}^\circ\nu_d(A) = \infty$ if $\nu_d(A) \notin \mathbb{F}^*\mathbb{R}$, induces the **Loeb measure** λ_d on X, defined on (the completion of) the σ-algebra generated by internal subsets of X (*a la* Caratheodory)
- $S \subseteq X$ is λ_d-*measurable* with finite measure $\lambda_d(S)$ iff
  $$\sup\{{}^\circ\nu_d(A);\ A \subseteq S,\ A \text{ is internal}\} = \inf\{{}^\circ\nu_d(B);\ S \subseteq B \subseteq X,\ B \text{ is internal}\}$$
  and both are finite

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Every regular complex-valued Borel measure with finite variation $\mu \in \mathrm{M}(\mathbf{X})$ is obtained in this way from some $g \in \mathcal{M}(X, X_{\mathrm{f}}, d)$; g is called **lifting** of μ w.r.t. X_{f}, E, d.

PvK Duality & FT in HF Ambience 1

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- $G^\flat = G_\mathrm{f}/G_0$ is abelian iff $[G_\mathrm{f}, G_\mathrm{f}] \subseteq G_0$, i.e.,
  $\forall\, x, y \in G_\mathrm{f} : [x, y] = xyx^{-1}y^{-1} \in G_0$

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- $K \subseteq j(F) + U = \bigcup_{a\in F} j(a) + U$
- $\forall\, a, b \in F \;:\; j(a), j(b), j(a+b) \in K \;\Rightarrow$
  $\Rightarrow\; j(a) + j(b) - j(a+b) \in U$

PvK Duality & FT in HF Ambience 3

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Can the dual group $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_\mathrm{f}/G_0}$ be described in terms of some group triplet, canonically related to the original triplet (G, G_0, G_f)?

PvK Duality & FT in HF Ambience 4

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$\widehat{G} = {}^*\mathrm{Hom}(G, {}^*\mathbb{T}) \left(\cong G \cong \widehat{\widehat{G}}\right)$ – internal dual group of G: all *internal* homomorphisms $\gamma : G \to {}^*\mathbb{T}$

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infinitesimal annihilators of arbitrary sets $X \subseteq G$, $\Gamma \subseteq \widehat{G}$

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- $\left(\widehat{G}, G_{\mathrm{f}}^{\curlywedge}, G_0^{\curlywedge}\right)$ – *dual group triplet* of (G, G_0, G_{f})

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What's the relation between the observable trace $\widehat{G}^\flat = G_0^{\dashv}/G_f^{\dashv}$ of the dual triplet $\left(\widehat{G}, G_f^{\dashv}, G_0^{\dashv}\right)$ and the dual $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_f/G_0}$ of the observable trace of the original triplet (G, G_0, G_f) ?

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Example:

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$$G^\flat = G_\mathrm{f}/G_0 \cong \begin{cases} \mathbb{Z} & \text{if } d \not\approx 0, \text{ as } G_0 = \{0\}, \\ \mathbb{T} & \text{if } d \approx 0,\ Kd < \infty, \text{ as } G_\mathrm{f} = G, \\ \mathbb{R} & \text{if } d \approx 0,\ Kd \notin \mathbb{F}^*\mathbb{R} \end{cases}$$

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Example: $\widehat{G}^\flat = G_0^{\perp}/G_{\mathrm{f}}^{\perp} \cong \widehat{G_{\mathrm{f}}/G_0} = \widehat{G^\flat}$

$K \in {}^*\mathbb{N} \setminus \mathbb{N},\ N = 2K+1,\ 0 < d < \infty,\ Kd \not\approx 0$

$G = \{nd;\ -K \le n \le K\}$ with $+ \bmod Nd$, $(G,+) \cong (\mathbb{Z}_N, +)$

$G_0 = G \cap \mathbb{I}^*\mathbb{R} = \{x \in G;\ x \approx 0\}$

$G_{\mathrm{f}} = G \cap \mathbb{F}^*\mathbb{R} = \{x \in G;\ |x| < \infty\}$

$$G^\flat = G_{\mathrm{f}}/G_0 \cong \begin{cases} \mathbb{Z} & \text{if } d \not\approx 0, \text{ as } G_0 = \{0\}, \\ \mathbb{T} & \text{if } d \approx 0,\ Kd < \infty, \text{ as } G_{\mathrm{f}} = G, \\ \mathbb{R} & \text{if } d \approx 0,\ Kd \notin \mathbb{F}^*\mathbb{R} \end{cases}$$

$\hat{d} = (Nd)^{-1},\ 0 < \hat{d} < \infty,\ K\hat{d} \not\approx 0$

$\widehat{G} = \{n\hat{d};\ -K \le n \le K\}$ with $+ \bmod N\hat{d}$, $(\widehat{G},+) \cong (\mathbb{Z}_N, +)$

$G_{\mathrm{f}}^{\perp} = \widehat{G} \cap \mathbb{I}^*\mathbb{R} = \{y \in \widehat{G};\ y \approx 0\}$

$G_0^{\perp} = \widehat{G} \cap \mathbb{F}^*\mathbb{R} = \{y \in \widehat{G};\ |y| < \infty\}$

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Example: $\widehat{G}^\flat = G_0^\perp / G_f^\perp \cong \widehat{G_f/G_0} = \widehat{G^\flat}$

$K \in {}^*\mathbb{N} \setminus \mathbb{N},\ N = 2K+1,\ 0 < d < \infty,\ Kd \not\approx 0$

$G = \{nd;\ -K \le n \le K\}$ with $+ \bmod Nd,\quad (G,+) \cong (\mathbb{Z}_N,+)$

$G_0 = G \cap \mathbb{I}^*\mathbb{R} = \{x \in G;\ x \approx 0\}$

$G_f = G \cap \mathbb{F}^*\mathbb{R} = \{x \in G;\ |x| < \infty\}$

$$G^\flat = G_f/G_0 \cong \begin{cases} \mathbb{Z} & \text{if } d \not\approx 0, \text{ as } G_0 = \{0\}, \\ \mathbb{T} & \text{if } d \approx 0,\ Kd < \infty, \text{ as } G_f = G, \\ \mathbb{R} & \text{if } d \approx 0,\ Kd \notin \mathbb{F}^*\mathbb{R} \end{cases}$$

$\hat{d} = (Nd)^{-1},\ 0 < \hat{d} < \infty,\ K\hat{d} \not\approx 0$

$\widehat{G} = \{n\hat{d};\ -K \le n \le K\}$ with $+ \bmod N\hat{d},\ (\widehat{G},+) \cong (\mathbb{Z}_N,+)$

$G_f^\perp = \widehat{G} \cap \mathbb{I}^*\mathbb{R} = \{y \in \widehat{G};\ y \approx 0\}$

$G_0^\perp = \widehat{G} \cap \mathbb{F}^*\mathbb{R} = \{y \in \widehat{G};\ |y| < \infty\}$

$\widehat{G}^\flat = G_0^\perp / G_f^\perp \cong$

PvK Duality & FT in HF Ambience 5

Example: $\widehat{G}^\flat = G_0^\perp / G_f^\perp \cong \widehat{G_f/G_0} = \widehat{G^\flat}$

$K \in {}^*\mathbb{N} \smallsetminus \mathbb{N},\ N = 2K+1,\ 0 < d < \infty,\ Kd \not\approx 0$

$G = \{nd;\ -K \le n \le K\}$ with $+ \bmod Nd$, $\quad (G,+) \cong (\mathbb{Z}_N,+)$

$G_0 = G \cap \mathbb{I}^*\mathbb{R} = \{x \in G;\ x \approx 0\}$

$G_f = G \cap \mathbb{F}^*\mathbb{R} = \{x \in G;\ |x| < \infty\}$

$$G^\flat = G_f/G_0 \cong \begin{cases} \mathbb{Z} & \text{if } d \not\approx 0, \text{ as } G_0 = \{0\}, \\ \mathbb{T} & \text{if } d \approx 0,\ Kd < \infty, \text{ as } G_f = G, \\ \mathbb{R} & \text{if } d \approx 0,\ Kd \notin \mathbb{F}^*\mathbb{R} \end{cases}$$

$\hat{d} = (Nd)^{-1},\ 0 < \hat{d} < \infty,\ K\hat{d} \not\approx 0$

$\widehat{G} = \{n\hat{d};\ -K \le n \le K\}$ with $+ \bmod N\hat{d}$, $\ (\widehat{G},+) \cong (\mathbb{Z}_N,+)$

$G_f^\perp = \widehat{G} \cap \mathbb{I}^*\mathbb{R} = \{y \in \widehat{G};\ y \approx 0\}$

$G_0^\perp = \widehat{G} \cap \mathbb{F}^*\mathbb{R} = \{y \in \widehat{G};\ |y| < \infty\}$

$$\widehat{G}^\flat = G_0^\perp / G_f^\perp \cong \begin{cases} \mathbb{T} & \text{if } d \not\approx 0, \text{ as } G_0^\perp = \widehat{G}, \\ \mathbb{Z} & \text{if } d \approx 0,\ Kd < \infty, \text{ as } G_f^\perp = \{0\}, \\ \mathbb{R} & \text{if } d \approx 0,\ Kd \notin \mathbb{F}^*\mathbb{R} \end{cases}$$

PvK Duality & FT in HF Ambience 6

PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\curlywedge}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\curlywedge}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$\begin{array}{ccccc} G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm{f}} & \xrightarrow[\circ]{} & \mathbf{G} \\ {\scriptstyle\gamma}\downarrow & & & & \downarrow{\scriptstyle\gamma^\flat} \\ {}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T} \end{array}$$

PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\curlywedge}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$
\begin{array}{ccccc}
G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm{f}} & \xrightarrow[\circ]{} & \mathbf{G} \\
\gamma\big\downarrow & & & & \big\downarrow\gamma^\flat \\
{}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T}
\end{array}
$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\curlywedge} \to \widehat{\mathbf{G}}$

PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\perp}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$\begin{array}{ccccc} G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm{f}} & \xrightarrow[\circ]{} & \mathbf{G} \\ \gamma\big\downarrow & & & & \big\downarrow\gamma^\flat \\ {}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T} \end{array}$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\perp} \to \widehat{\mathbf{G}}$
- with image $\{\gamma^\flat;\ \gamma \in G_0^{\perp}\}$ and kernel $G_{\mathrm{f}}^{\perp}$

PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\downarrow}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_\mathrm{f}$:

$$
\begin{array}{ccccc}
G & \xleftarrow[\mathrm{id}]{} & G_\mathrm{f} & \xrightarrow[\circ]{} & \mathbf{G} \\
{\scriptstyle\gamma}\downarrow & & & & \downarrow{\scriptstyle\gamma^\flat} \\
{}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T}
\end{array}
$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\downarrow} \to \widehat{\mathbf{G}}$
- with image $\left\{\gamma^\flat;\ \gamma \in G_0^{\downarrow}\right\}$ and kernel $G_\mathrm{f}^{\downarrow}$
- it induces injective homomorphism $G_0^{\downarrow}/G_\mathrm{f}^{\downarrow} \to \widehat{\mathbf{G}}$

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Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\perp}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$\begin{array}{ccccc} G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm{f}} & \xrightarrow[\circ]{} & \mathbf{G} \\ {\scriptstyle\gamma}\big\downarrow & & & & \big\downarrow{\scriptstyle\gamma^\flat} \\ {}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T} \end{array}$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\perp} \to \widehat{\mathbf{G}}$
- with image $\left\{\gamma^\flat;\ \gamma \in G_0^{\perp}\right\}$ and kernel $G_{\mathrm{f}}^{\perp}$
- it induces injective homomorphism $G_0^{\perp}/G_{\mathrm{f}}^{\perp} \to \widehat{\mathbf{G}}$
- this canonic mapping is isomorphism of $\widehat{G}^\flat = G_0^{\perp}/G_{\mathrm{f}}^{\perp}$ onto closed subgroup of $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_{\mathrm{f}}/G_0}$

PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\downarrow}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$\begin{array}{ccccc} G & \xleftarrow[\text{id}]{\quad\quad} & G_{\mathrm{f}} & \xrightarrow[\circ]{\quad\quad} & \mathbf{G} \\ \gamma\big\downarrow & & & & \big\downarrow\gamma^\flat \\ {}^*\mathbb{T} & \xrightarrow{\ \circ\ } & {}^*\mathbb{T}/\approx & \xleftarrow{\ \text{id}\ } & \mathbb{T} \end{array}$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\downarrow} \to \widehat{\mathbf{G}}$
- with image $\left\{\gamma^\flat;\ \gamma \in G_0^{\downarrow}\right\}$ and kernel $G_{\mathrm{f}}^{\downarrow}$
- it induces injective homomorphism $G_0^{\downarrow}/G_{\mathrm{f}}^{\downarrow} \to \widehat{\mathbf{G}}$
- this canonic mapping is isomorphism of $\widehat{G}^\flat = G_0^{\downarrow}/G_{\mathrm{f}}^{\downarrow}$ onto closed subgroup of $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_{\mathrm{f}}/G_0}$
- non-S-continuous internal characters $\gamma \in \widehat{G} \setminus G_0^{\downarrow}$ correspond neither to non-continuous characters of $\mathbf{G}$, nor even to mappings $\mathbf{G} \to \mathbb{T}$

PvK Duality & FT in HF Ambience 7

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\dashv}/G_{\mathrm{f}}^{\dashv} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\curvearrowleft}/G_{\mathrm{f}}^{\curvearrowleft} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^\flat$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^\flat$ for some $\gamma \in G_0^{\curvearrowleft}$

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\perp}/G_{\mathrm{f}}^{\perp} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^{\flat}$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^{\flat}$ for some $\gamma \in G_0^{\perp}$
- by PvK duality this is equivalent to $G_0^{\perp\perp} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^{\flat};\ \gamma \in G_0^{\perp}\}$ separate points in $\mathbf{G}$ (density follows)

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\wedge}/G_{\mathrm{f}}^{\wedge} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^\flat$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^\flat$ for some $\gamma \in G_0^{\wedge}$
- by PvK duality this is equivalent to $G_0^{\wedge\wedge} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^\flat;\ \gamma \in G_0^{\wedge}\}$ separate points in $\mathbf{G}$ (density follows)
- I proved a bit more [PZ, June – July 2012]: $G_0^{\wedge\wedge} = G_0$

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\curlywedge}/G_{\mathrm{f}}^{\curlywedge} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^{\flat}$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^{\flat}$ for some $\gamma \in G_0^{\curlywedge}$
- by PvK duality this is equivalent to $G_0^{\curlywedge\curlywedge} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^{\flat};\ \gamma \in G_0^{\curlywedge}\}$ separate points in $\mathbf{G}$ (density follows)
- I proved a bit more [PZ, June – July 2012]: $G_0^{\curlywedge\curlywedge} = G_0$
- $G_{\mathrm{f}}^{\curlywedge\curlywedge} = G_{\mathrm{f}} + G_0^{\curlywedge\curlywedge} = G_{\mathrm{f}}$, i.e., the dual triplet of $\left(\widehat{G}, G_{\mathrm{f}}^{\curlywedge}, G_0^{\curlywedge}\right)$ is (G, G_0, G_{f})

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):
The canonic mapping $G_0^{\curlywedge}/G_{\mathrm{f}}^{\curlywedge} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^\flat$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^\flat$ for some $\gamma \in G_0^{\curlywedge}$
- by PvK duality this is equivalent to $G_0^{\curlywedge\curlywedge} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^\flat;\ \gamma \in G_0^{\curlywedge}\}$ separate points in $\mathbf{G}$ (density follows)
- I proved a bit more [PZ, June – July 2012]: $G_0^{\curlywedge\curlywedge} = G_0$
- $G_{\mathrm{f}}^{\curlywedge\curlywedge} = G_{\mathrm{f}} + G_0^{\curlywedge\curlywedge} = G_{\mathrm{f}}$, i.e., the dual triplet of $\left(\widehat{G}, G_{\mathrm{f}}^{\curlywedge}, G_0^{\curlywedge}\right)$ is (G, G_0, G_{f})
- *methods*: NSA + Harmonic An. + Additive Combinatorics (G. Freiman, B. Green, I. Ruzsa, T. Tao, V. Vu, ...): analysis of Bohr sets and **spectral sets** $\mathcal{S}_t(f) = \left\{\gamma \in \widehat{G};\ |\widehat{f}(\gamma)| \geq t\,\|f\|_1\right\}$ $\left(f \in \mathbb{C}^G,\ {}^*\mathbb{C}^G,\ t \in [0,1]\right)$

PvK Duality & FT in HF Ambience 8

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\perp} = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\dashv} = \{1_G\}$ (trivial character), hence $H^{\dashv\dashv} = G$.

The equality $G_0^{\dashv\dashv} = G_0$, in fact the inclusion $G_0^{\dashv\dashv} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\downarrow} = \{1_G\}$ (trivial character), hence $H^{\downarrow\downarrow} = G$.

The equality $G_0^{\downarrow\downarrow} = G_0$, in fact the inclusion $G_0^{\downarrow\downarrow} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\downarrow} = \{1_G\}$ (trivial character), hence $H^{\downarrow\downarrow} = G$.

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If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\downarrow} = \{1_G\}$ (trivial character), hence $H^{\downarrow\downarrow} = G$.

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Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\downarrow} = \{1_G\}$ (trivial character), hence $H^{\downarrow\downarrow} = G$.

The equality $G_0^{\downarrow\downarrow} = G_0$, in fact the inclusion $G_0^{\downarrow\downarrow} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

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If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

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There are proper nontrivial subgroups $H \le G$ s.t. $H^{\curlywedge} = \{1_G\}$ (trivial character), hence $H^{\curlywedge\curlywedge} = G$.

The equality $G_0^{\curlywedge\curlywedge} = G_0$, in fact the inclusion $G_0^{\curlywedge\curlywedge} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \ge 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \le j \le n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \le q_j$$

then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

- by PvK duality: $\forall\, \alpha, \beta\ \forall\, \mathbf{q}\ \forall\, G\ \forall\, (A_j)\ \exists\, n\ :\ \ldots \Rightarrow \ldots$

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \le G$ s.t. $H^{\downarrow} = \{1_G\}$ (trivial character), hence $H^{\downarrow\downarrow} = G$.

The equality $G_0^{\downarrow\downarrow} = G_0$, in fact the inclusion $G_0^{\downarrow\downarrow} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \ge 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \le j \le n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \le q_j$$

then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

- by PvK duality: $\forall\,\alpha, \beta\ \forall\,\mathbf{q}\ \forall\,G\ \forall\,(A_j)\ \exists\,n\ :\ \ldots \Rightarrow \ldots$
- gained uniformity: $\forall\,\alpha, \beta\ \forall\,\mathbf{q}\ \exists\,n\ \forall\,G\ \forall\,(A_j)\ :\ \ldots \Rightarrow \ldots$

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\perp} = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

The equality $G_0^{\perp\perp} = G_0$, in fact the inclusion $G_0^{\perp\perp} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

- by PvK duality: $\forall\, \alpha, \beta\ \forall\, \mathbf{q}\ \forall\, G\ \forall\, (A_j)\ \exists\, n\ :\ \ldots \Rightarrow \ldots$
- gained uniformity: $\forall\, \alpha, \beta\ \forall\, \mathbf{q}\ \exists\, n\ \forall\, G\ \forall\, (A_j)\ :\ \ldots \Rightarrow \ldots$
- no estimate for $n = n(\alpha, \beta, \mathbf{q})$

PvK Duality & FT in HF Ambience 9

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^{\perp\!\!\!\perp}/G_{\mathrm{f}}^{\perp\!\!\!\perp} \to \widehat{G_{\mathrm{f}}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^{\perp\!\!\!\!\sim}/G_{\mathrm{f}}^{\perp\!\!\!\!\sim} \to \widehat{G_{\mathrm{f}}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^{\perp}/G_{\mathrm{f}}^{\perp} \to \widehat{G_{\mathrm{f}}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^\perp / G_f^\perp \to \widehat{G_f / G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^\infty$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

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PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^{\perp\!\!\!\perp}/G_{\mathrm{f}}^{\perp\!\!\!\perp} \to \widehat{G_{\mathrm{f}}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

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"partial δ-homomorphism" $g : A \to \mathbb{T}$ means:

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PvK Duality & FT in HF Ambience 10

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PvK Duality & FT in HF Ambience 10

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I gave a more direct and clear proof of **GC2** by similar methods like those in **GC1**.

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PvK Duality & FT in HF Ambience 12

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- S^∞-continuous is just S-continuous

PvK Duality & FT in HF Ambience 13

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PvK Duality & FT in HF Ambience 13

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PvK Duality & FT in HF Ambience 13

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PvK Duality & FT in HF Ambience 14

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Smoothness-and-Decay Principle.

PvK Duality & FT in HF Ambience 14

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PvK Duality & FT in HF Ambience 14

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PvK Duality & FT in HF Ambience 14

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PvK Duality & FT in HF Ambience 14

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TYFYAP

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