Pontryagin-van Kampen Duality and Fourier Transform in Hyperfinite Ambience

Gordon’s Conjectures

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Pontryagin-van Kampen Duality

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Nonstandard Analysis

PvK Duality & FT in HF Ambience

\textit{Pontryagin-van Kampen Duality 1}
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Pontryagin-van Kampen Duality

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$${\mathcal{B}}_\alpha(\Gamma) = \{ x \in G; \ \forall \gamma \in \Gamma : |\arg \gamma(x)| \leq \alpha \}$$
Pontryagin-van Kampen Duality

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**Pontryagin-van Kampen Duality 2**

\( \hat{G} \) is an LCA group with pointwise multiplication and compact-open topology – dual group of \( G \).

\[ \{ B_\alpha(A); 0 \in A \subseteq G \text{ is compact and } 0 < \alpha < \frac{2\pi}{3} \} \]

Examples.

\( \hat{G} \times H \cong \hat{G} \times \hat{H} \)

\( \hat{\mathbb{Z}}_n \cong \mathbb{Z}_n \):

\( a \in \mathbb{Z}_n \) corresponds to the character \( k \mapsto e^{2\pi i ak/n} \).

\( \hat{G} \cong G \) for each finite abelian group \( G \).

\( \hat{T} \cong \mathbb{Z} \):

\( c \in \mathbb{T} \) corresponds to the character \( n \mapsto c^n \).

\( \hat{\mathbb{R}} \cong \mathbb{R} \):

\( a \in \mathbb{R} \) corresponds to the character \( t \mapsto e^{iat} \).
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- \( \hat{G} \) is an **LCA group** with pointwise multiplication and compact-open topology – **dual group** of \( G \)
Pontryagin-van Kampen Duality 2

- $\hat{G}$ is an LCA group with pointwise multiplication and compact-open topology – dual group of $G$
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Pontryagin-van Kampen Duality

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- \( \widehat{G \times H} \cong \hat{G} \times \hat{H} \)
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Pontryagin-van Kampen Duality

Pontryagin-van Kampen Duality 3

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Pontryagin-van Kampen Duality

The natural mapping
\[ G \to \hat{\hat{G}}, \]
assigning to \( x \in G \) the character \( \hat{x} : \hat{\hat{G}} \to \mathbb{T} \), given by
\[ \hat{x}(\gamma) = \gamma(x) \]
for \( \gamma \in \hat{\hat{G}} \), is isomorphism of topological groups
\[ G \cong \hat{\hat{G}}. \]

Consequences.

• \( G \) is discrete (compact) iff \( \hat{\hat{G}} \) is compact (discrete)

• \( G \) is metrizable (\( \sigma \)-compact) iff \( \hat{\hat{G}} \) is \( \sigma \)-compact (metrizable)

• Subgroup \( D \leq \hat{\hat{G}} \) is dense in \( \hat{\hat{G}} \) iff the characters \( \gamma \in D \) separate points in \( G \)

• \( G \) is connected iff \( \hat{\hat{G}} \) has no nontrivial compact subgroups

• \( G \) is totally disconnected iff each \( \gamma \in \hat{\hat{G}} \) generates a relatively compact subgroup
Pontryagin-van Kampen Duality theorem.
Pontryagin-van Kampen Duality 3

Pontryagin-van Kampen duality theorem.

The natural mapping \( G \rightarrow \hat{G} \), assigning to \( x \in G \) the character \( x : \hat{G} \rightarrow \mathbb{T} \), given by \( x(\gamma) = \gamma(x) \) for \( \gamma \in \hat{G} \), is isomorphism of topological groups \( G \cong \hat{G} \).
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Fourier Transform 1
$m_G$, $m_{\hat{G}}$ denote Haar measures on $G$, $\hat{G}$, resp.
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- \( \hat{f}(\gamma) = \int f \cdot \overline{\gamma} \, dm_G = \int f(x) \overline{\gamma(x)} \, dm_G(x) \)
  for \( f \in L^1(G) = L^1(G, m_G), \quad \gamma \in \hat{G} \)
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- $\hat{f}$ is the Fourier transform of $f$, 

$\| \hat{f} \|_{\infty} \leq \| f \|_1$
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- \(\hat{f}\) is the Fourier transform of \(f\), \(\|\hat{f}\|_\infty \leq \|f\|_1\)
- \(f \mapsto \mathcal{F}(f) = \hat{f}\) defines bounded linear operator \(\mathcal{F} : L^1(G) \rightarrow C_0(\hat{G})\)
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- $\mathcal{F} = \mathcal{F}_G$ is called the **Fourier transform** on $G$
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- $L^1(G)$ is associative algebra under convolution
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- $\mathcal{F} = \mathcal{F}_G$ is called the **Fourier transform** on $G$
- $L^1(G)$ is associative algebra under convolution
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In what follows $1 \leq p, q \leq \infty$, \[ \frac{1}{p} + \frac{1}{q} = 1. \]

For $1 \leq p < \infty$, $L^1(G) \cap L^p(G)$ is dense in $L^p(G)$ w.r.t. the $L^p$-norm.

For $1 \leq p \leq 2$, the restricted Fourier transform $F : L^1(G) \cap L^p(G) \to L^q(\hat{G})$ satisfies \[ \| \hat{f} \|_q \leq \| f \|_p \] for $f \in L^1(G) \cap L^p(G)$.

It extends to the Fourier transform $F : L^p(G) \to L^q(\hat{G})$.

For $p = 2$ we get the Fourier-Plancherel transform which is an isometric isomorphism of Hilbert spaces $F : L^2(G) \to L^2(\hat{G})$.

Proper normalization of Haar measures $m_G, m_{bG}$ ensures the Plancherel identity and Fourier inversion formula:

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\( M(G) \) is the Banach space of all complex-valued regular Borel measures \( \mu \) on \( G \) with bounded total variation \( \| \mu \| \).
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Fourier Transform 5
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For finite \( G \) all the above spaces coincide with \( \mathbb{C}^G \), the scalar product is everywhere defined, and the Fourier inversion formula holds.
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Isn’t there some “universal extension” of all the spaces \( L^{p}(G) \) \((1 \leq p \leq 2)\) and \( M(G) \), and a uniform scheme defining the Fourier transform on this extension, covering all the particular cases, like if \( G \) were finite?
Nonstandard Analysis 1

Nonstandard analysis offers solution and additional insights. NSA is a method based on application of mathematical logic to other parts of mathematics, invented by A. Robinson (1960s).

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P. Vopěnka: Odmítnutí Newtonova a Lebnizova pojetí infinitesimálního kalkulu matematiky 19. a 20. století – vyvolané ať již jejich neochotou či neschopností domyslet a dotvořit základní pojmy, o než se původní pojetí tohoto kalkulu opíralo – bylo jedním z největších omylů nejen matematiky, ale evropské vědy vůbec.
Nonstandard Analysis 2

Merits and contributions of NSA (among other things):

• rehabilitation of the original infinitesimal calculus with infinitely small and infinitely big numerical magnitudes

• continuity:
  \[ f(x + d) \approx f(x) \text{ for } d \approx 0 \]

• derivative:
  \[ f'(x) \approx \frac{f(x + d) - f(x)}{d}, \text{ where } 0 \neq d \approx 0 \]

• integral:
  \[ \int_a^b f(x) \, dx \approx \sum_{n=1}^{\infty} f(x_k) \, d, \quad a + (k-1)d \leq x_k \leq a + kd, \] for "infinite" \( n \in \mathbb{N} \),
  \( d = \frac{b-a}{n} \approx 0 \)

• extension of all domains of mathematical objects by abundance of new ideal elements

• extended domains have the same mathematical properties w.r.t. original first-order language (transfer principle)

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- \( f'(x) \approx \frac{f(x+d) - f(x)}{d} \), where \( 0 \neq d \approx 0 \)
- \( \int_{a}^{b} f(x) \, dx \approx \sum_{k=1}^{n} f(x_k) \, d \), \( a + (k - 1)d \leq x_k \leq a + kd \), for “infinite” \( n \in \mathbb{N}, d = (b - a)/n \approx 0 \)
- extension of all domains of mathematical objects by abundance of new ideal elements
- extended domains have the same mathematical properties w.r.t. original first-order language (transfer principle)
Nonstandard Analysis 2

Merits and contributions of NSA (among other things):

- rehabilitation of the original infinitesimal calculus with infinitely small and infinitely big numerical magnitudes
- continuity: \( f(x + d) \approx f(x) \) for \( d \approx 0 \)
- \( f'(x) \approx \frac{f(x+d) - f(x)}{d} \), where \( 0 \neq d \approx 0 \)
- \( \int_a^b f(x) \, dx \approx \sum_{k=1}^{n} f(x_k) \, d \), \( a + (k - 1)d \leq x_k \leq a + kd \), for “infinite” \( n \in \mathbb{N} \), \( d = (b - a)/n \approx 0 \)
- extension of all domains of mathematical objects by abundance of new ideal elements
- extended domains have the same mathematical properties w.r.t. original first-order language (transfer principle)
- every consistent “not too big” system of standard formulas is satisfied by some object (saturation)
Nonstandard Analysis 3
Nonstandard Analysis 3

In particular:
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- $^*\mathbb{N}$ – hypernatural numbers: $\mathbb{N} < ^*\mathbb{N}$, $^*\mathbb{N} \setminus \mathbb{N} \neq \emptyset$
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- \( \mathbb{N} \) – hypernatural numbers: \( \mathbb{N} \prec \mathbb{N}, \mathbb{N} \setminus \mathbb{N} \neq \emptyset \)

- \( \mathbb{R} \) – hyperreal numbers: \( \mathbb{R} \prec \mathbb{R}, \mathbb{R} \setminus \mathbb{R} \neq \emptyset \)

  \[ \mathbb{F}\mathbb{R} = \{ x \in \mathbb{R}; \exists \ r \in \mathbb{R}, \ r > 0 : |x| \leq r \} \]
  \[ \mathbb{I}\mathbb{R} = \{ x \in \mathbb{R}; \forall \ r \in \mathbb{R}, \ r > 0 : |x| \leq r \} \]

  finite and infinitesimal hyperreals, resp.

- \( \mathbb{C} \) – hypercomplex numbers: \( \mathbb{C} \prec \mathbb{C}, \mathbb{C} \setminus \mathbb{C} \neq \emptyset \)

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In particular:

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- **C** – hypercomplex numbers: \( \mathbb{C} \prec \mathcal{C}, \mathcal{C} \setminus \mathbb{C} \neq \emptyset \)
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  finite and infinitesimal hypercomplex numbers, resp.
- for each finite hyperreal or hypercomplex number \( x \) there is unique real or complex number \( \circ x = \text{st} \ x \), called *shadow* or *standard part* of \( x \), such that \( x \approx \circ x \)
Nonstandard Analysis 3

In particular:

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  finite and infinitesimal hypercomplex numbers, resp.
- for each finite hyperreal or hypercomplex number \( x \) there is unique real or complex number \( \circ x = \text{st} x \), called shadow or standard part of \( x \), such that \( x \approx \circ x \)
- \( F^*\mathbb{R}/I^*\mathbb{R} \cong \mathbb{R} \), \( F^*\mathbb{C}/I^*\mathbb{C} \cong \mathbb{C} \)
Nonstandard Analysis 4

In general:

• every "standard" mathematical object $M$ has its "nonstandard" extension $\ast M \succ M$; the embedding $M \rightarrow \ast M$ is onto iff $M$ is finite.

• the whole universe $U$ of "standard" mathematical objects is embedded into the universe $\ast U$ of "nonstandard" (internal) objects.

• every standard set $M$ can be embedded into a hyperfinite set $H \subseteq \ast M$.

• every standard vector space $V$ over a field $K$ can be embedded into a hyperfinite dimensional vector space $H \subseteq \ast V$ over $\ast K$. 
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Nonstandard Analysis 4

In general:

- every “standard” mathematical object $M$ has its “nonstandard” extension $\star M \supset M$
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- the whole universe $\mathcal{U}$ of “standard” mathematical objects is embedded into the universe $\star \mathcal{U}$ of “nonstandard” (internal) objects (standard, internal and external objects)
- every standard set $M$ can be embedded into a \textit{hyperfinitely set} $H \subseteq \star M$
In general:

- every “standard” mathematical object \( M \) has its “nonstandard” extension \( *M \succ M \)
  the embedding \( M \rightarrow *M \) is onto iff \( M \) is finite
- the whole universe \( \mathcal{U} \) of “standard” mathematical objects is embedded into the universe \( *\mathcal{U} \) of “nonstandard” (internal) objects (standard, internal and external objects)
- every standard set \( M \) can be embedded into a \textit{hyperfinite set} \( H \subseteq *M \)
- every standard vector space \( V \) over a field \( K \) can be embedded into a \textit{hyperfinite dimensional vector space} \( H \subseteq *V \) over \( *K \)
On the other hand, not every standard group can be embedded into a hyperfinite group. In general, algebraic structure can be an obstacle to embedability into hyperfinite objects. However, every standard abelian group can be embedded into a hyperfinite abelian group.

D. Zeilberger: Continuous analysis and geometry are just degenerate approximations to the discrete world [...] While discrete analysis is conceptually simpler (and truer) than continuous analysis, technically it is (usually) much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics [...].
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Nonstandard Analysis 6
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Nonstandard Analysis 6

Every completely regular topological space $X$ can be represented by a triplet $(X, E, X_f)$, where

- $X$ is an ambient hyperfinite set
- $E$: equivalence relation of infinitesimal nearness on $X$, 

$X_f$ is the set of "finite" or "nearstandard", i.e., accessible elements of $X$; $y \approx x \in X_f \Rightarrow y \in X_f$, i.e., $X_f = E[X_f]$.

$X \sim = X_f / E = X^\flat$ – the nonstandard hull or observable trace of $(X, E, X_f)$.
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And $X_f$ and $E$ are external sets; $X_f$ is union and $E$ is intersection of “not too many” internal sets ($\Sigma^0_1$ and $\Pi^0_1$).
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Nonstandard Analysis 7
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Topology of both $X$ and $X$ is more intuitively described in terms of $\approx$. 
Nonstandard Analysis 7

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- $Y \subseteq X$ is open iff $\forall x \in Y \exists \text{int } A : \text{mon}(x) \subseteq A \subseteq Y$
Nonstandard Analysis 7

Topology of both $X$ and $\mathbf{X}$ is more intuitively described in terms of $\approx$.

- $Y \subseteq X$ is open iff $\forall x \in Y \exists \text{int } A : \text{mon}(x) \subseteq A \subseteq Y$
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- $\mathbf{X}$ is discrete iff $E$ is internal on $X_f$ (then $E = \text{Id}_X$, w.l.o.g.)
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- $\mathbf{X}$ is locally compact iff every hyperfinite set $H \subseteq X_f$ of pairwise discernible elements is finite
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- $\mathbf{X}$ is locally compact iff every hyperfinite set $H \subseteq X_f$ of pairwise discernible elements is finite
- $\mathbf{X}$ is compact iff, additionally, $X_f$ is internal (then $X_f = X$, w.l.o.g.)
Nonstandard Analysis

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- $X$ is locally compact iff every hyperfinite set $H \subseteq X_f$ of pairwise discernible elements is finite
- $X$ is compact iff, additionally, $X_f$ is internal (then $X_f = X$, w.l.o.g.)
- for $X$ locally compact, compact subsets of $X$ are exactly $A^\flat = \{ \circ a; \ a \in A \}$ for internal $A \subseteq X_f$ (“pushing-down” $A$)
Nonstandard Analysis 8
Nonstandard Analysis 8

Internal function $f : X \to *\mathbb{C}$ represents a function $f : X \to \mathbb{C}$ only if $f$ is $S$-continuous and finite on $X_f$:
Nonstandard Analysis

Internal function $f : X \rightarrow \ast \mathbb{C}$ represents a function $f : X \rightarrow \mathbb{C}$ only if $f$ is $S$-continuous and finite on $X_f$:

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Internal function $f : X \to \ast \mathbb{C}$ represents a function $f : X \to \mathbb{C}$ only if $f$ is $S$-continuous and finite on $X_f$:

- $x \approx y \Rightarrow f(x) \approx f(y)$ for $x, y \in X_f$
- $|f(x)| < \infty$, i.e., $f(x) \in F^* \mathbb{C}$ for $x \in X_f$
Internal function $f : X \to *\mathbb{C}$ represents a function $\mathbf{f} : \mathbf{X} \to \mathbb{C}$ only if $f$ is $S$-continuous and finite on $X_f$:

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- $f \in C(X, E, X_f)$ is called lifting of $f \in C(X)$
Internal function \( f : X \to \ast \mathbb{C} \) represents a function \( \mathbf{f} : X \to \mathbb{C} \) only if \( f \) is \( S \)-continuous and finite on \( X_f \):

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- then \( \mathbf{f}(\circ x) = \circ f(x) = \text{st} f(x) \) and \( \mathbf{f} \) is continuous
- \( f \in C(X, E, X_f) \) is called \textbf{lifting} of \( \mathbf{f} \in C(X) \)
- \( \mathbf{f} = f^\flat = \circ (f | X_f) \), \( \mathbf{f} \) is obtained by “pushing-down” \( f \)
Internal function $f : X \to \ast \mathbb{C}$ represents a function $f : X \to \mathbb{C}$ only if $f$ is $S$-continuous and finite on $X_f$:

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- $|f(x)| < \infty$, i.e., $f(x) \in \mathbb{F}^\ast \mathbb{C}$ for $x \in X_f$
- then $f(\circ x) = \circ f(x) = \text{st } f(x)$ and $f$ is continuous
- $f \in C(X, E, X_f)$ is called \textbf{lifting} of $f \in C(X)$
- $f = f^b = \circ (f \upharpoonright X_f)$, $f$ is obtained by “pushing-down” $f$
- $f \in C_u(X)$ iff $f$ has (everywhere) $S$-continuous lifting $f \in C(X, E)$
Internal function \( f : X \rightarrow \ast \mathbb{C} \) represents a function \( f : X \rightarrow \mathbb{C} \) only if \( f \) is \( S \)-continuous and finite on \( X_f \):

- \( x \approx y \Rightarrow f(x) \approx f(y) \) for \( x, y \in X_f \)
- \( |f(x)| < \infty \), i.e., \( f(x) \in F^* \mathbb{C} \) for \( x \in X_f \)
- then \( f(\circ x) = \circ f(x) = \text{st } f(x) \) and \( f \) is continuous
- \( f \in C(X, E, X_f) \) is called \textbf{lifting} of \( f \in C(X) \)
- \( f = f^b = \circ (f \upharpoonright X_f) \), \( f \) is obtained by “pushing-down” \( f \)
- \( f \in C_u(X) \) iff \( f \) has (everywhere) \( S \)-continuous lifting \( f \in C(X, E) \)
- \( f \in C_b(X) \) iff \( f \) has lifting \( f \in C_b(X, E, X_f) \), i.e., \( f \in C(X, E, X_f) \) and \( \|f\|_{\infty} = \max_{x \in X} |f(x)| < \infty \)
Internal function $f : X \to \ast\mathbb{C}$ represents a function $\mathbf{f} : \mathbf{X} \to \mathbb{C}$ only if $f$ is $S$-continuous and finite on $X_f$:

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- $|f(x)| < \infty$, i.e., $f(x) \in \mathbb{F}^\ast \mathbb{C}$ for $x \in X_f$
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- $f \in \mathcal{C}(X, E, X_f)$ is called lifting of $\mathbf{f} \in \mathcal{C}(\mathbf{X})$
- $\mathbf{f} = f^\flat = \circ (f \restriction X_f)$, $\mathbf{f}$ is obtained by “pushing-down” $f$
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- $\mathbf{f} \in \mathcal{C}_b(\mathbf{X})$ iff $\mathbf{f}$ has lifting $f \in \mathcal{C}_b(X, E, X_f)$, i.e., $f \in \mathcal{C}(X, E, X_f)$ and $\|f\|_\infty = \max_{x \in X} |f(x)| < \infty$
- $\mathbf{f} \in \mathcal{C}_0(\mathbf{X})$ iff $\mathbf{f}$ has lifting $f \in \mathcal{C}_0(X, E, X_f)$, i.e., $f \in \mathcal{C}_b(X, E, X_f)$ and $f(x) \approx 0$ for $x \in X \setminus X_f$
Nonstandard Analysis

Not all internal functions $f : X \to \ast \mathbb{C}$ represent standard functions $X \to \mathbb{C}$. Some of them represent standard objects of different nature: cosets of functions in Lebesgue $L^p$ spaces, measures, distributions, etc.
Nonstandard Analysis 9

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Nonstandard Analysis 10
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Loeb measure.
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$X$ is hyperfinite set, $d : X \to ^*\mathbb{R}$ is internal, $d(x) \geq 0$ for $x \in X$; typically, $d(x) = d$ is uniform (constant) on $X$.
Nonstandard Analysis 10

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- $A \mapsto {}^\circ \nu_d(A)$, where ${}^\circ \nu_d(A) = \infty$ if $\nu_d(A) \notin \mathbb{F}^*\mathbb{R}$, induces the **Loeb measure** $\lambda_d$ on $X$, defined on (the completion of) the $\sigma$-algebra generated by internal subsets of $X$ (*a la* Caratheodory)
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- $S \subseteq X$ is $\lambda_d$-measurable with finite measure $\lambda_d(S)$ iff $\sup\{\nabla \nu_d(A); A \subseteq S, A \text{ is internal}\} = \inf\{\nabla \nu_d(B); S \subseteq B \subseteq X, B \text{ is internal}\}$ and both are finite
Nonstandard Analysis 11
Every standard locally compact measurable space \((X, \mathcal{B}, m)\), s.t. ..., can be represented as the observable trace \(X = X_f/E\) of some “topological Loeb quadruplet” \((X, E, X_f, d)\), with hyperfinite \(X\) and uniform \(d\), in such a way that

- \(\nu_d(A) = d|A| < \infty\) for all internal \(A \subseteq X_f\)
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- \(\nu_d(A) = d|A| < \infty\) for all internal \(A \subseteq X_f\)
- then, for \(A \in \mathcal{B}\),
  \[
  m(A) = \lambda_d\{x \in X_f; \circ x \in A\}
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\[
\begin{array}{cccccc}
  X & \xleftarrow{id} & X_f & \xrightarrow{\circ} & X \\
  \downarrow f & & & & \downarrow f = f^b \\
  *\mathbb{C} & \xrightarrow{\circ} & *\mathbb{C}/ \approx & \xleftarrow{id} & \mathbb{C}
\end{array}
\]
Nonstandard Analysis 12
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Internal function \( f : X \to \ast \mathbb{C} \) is called \textbf{S-integrable} w.r.t. \( X_f \) and \( d \) (even for nonconstant \( d \)) if

\[
\|f\|_{1,d} = \sum_{x \in X} |f(x)| d(x) < \infty
\]

\[
\sum_{z \in \mathbb{Z}} |f(z)| d(z) \approx 0 \quad \text{for internal } \mathbb{Z} \subseteq X \setminus X_f
\]

\[
\sum_{a \in \Delta} |f(a)| d(a) \approx 0 \quad \text{for internal } \Delta \subseteq X \text{ s.t. } \nu d(A) \approx 0
\]

\[
\int_X f \circ f \, d\lambda \approx \sum_{x \in X} f(x) d(x)
\]

\( f \) is \textbf{S}-integrable if \( f^p \) is \textbf{S}-integrable (\( 1 \leq p < \infty \))
Nonstandard Analysis 12

Internal function $f : X \to \ast \mathbb{C}$ is called $S$-integrable w.r.t. $X_f$ and $d$ (even for nonconstant $d$) if

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Let, additionally, $(X, \mathcal{B}, m)$ be represented by $(X, E, X_f, d)$, and $f : X \to \mathbb{C}$. Then
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- \( f \) is \( S^p \)-integrable if \( f^p \) is \( S \)-integrable (\( 1 \leq p < \infty \))

Let, additionally, \( (X, \mathcal{B}, m) \) be represented by \( (X, E, X_f, d) \), and \( f : X \to \mathbb{C} \). Then

- \( f \in L^1(X, m) \) iff \( f \) has \( S \)-integrable lifting
- \( f \in L^p(X, m) \) iff \( f \) has \( S^p \)-integrable lifting
- \( L^p(X, X_f, d) = \{ f \in \ast \mathbb{C}^X ; f \text{ lifts some } f \in L^p(X, m) \} \)
If \( f \in L^1(X, X_f, \mu) \) is an \( S_p \)-integrable lifting of \( f \in L^1(X, \mu) \), then

\[
\int_X f \, d\mu = \int_X f \circ f \, d\lambda \approx \sum_{x \in X} f(x) \, d(x) \quad \|f\|_1 = \int_X |f| \, d\mu = \int_X f \circ |f| \, d\lambda \approx \sum_{x \in X} |f(x)| \, d(x) = \|f\|_1,
\]

Remarks.

Not every \( S_p \)-integrable function is in \( L^p(X, X_f, \mu) \)!

\( L^p(X, X_f, \mu) \) is dense in the subspace of \( S_p \)-integrable functions w.r.t. some "week topology".

\( L^p(X, X_f, \mu) \) is the closure of \( C_{00}(X, E, X_f) \) w.r.t. the \( p \)-norm

\[
\|f\|_{p, \mu} = \left( \sum_{x \in X} |f(x)|^p \right)^{1/p}.
\]
If $f \in \mathcal{L}^1(X, X_f, d)$ is $S$-integrable lifting of $f \in L^1(X, m)$ then

\[ \|f\|_1 = \int_X |f| \, dm = \int_X f \cdot |f| \, \lambda \, dm \approx \sum_{x \in X} |f(x)| \, d(x) = \|f\|_1, \]
If $f \in \mathcal{L}^1(X, X_f, d)$ is $S$-integrable lifting of $f \in L^1(X, m)$ then

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Nonstandard Analysis 13

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If $f \in \mathcal{L}^1(X, X_f, d)$ is $S$-integrable lifting of $f \in L^1(X, m)$ then

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\[
\|f\|_{p,d} = \left( \sum_{x \in X} |f(x)|^p \, d(x) \right)^{1/p}
\]
M(\mathbb{X},\mathbb{X}_f,d) consist of all internal functions \(g: \mathbb{X} \to \mathbb{C}^*\) s.t. (the third condition defining S-integrability is omitted)

\[ \|g\|_{1,d} = \sum_{x \in \mathbb{X}} |g(x)| d(x) < \infty \]

\sum_{z \in \mathbb{Z}} |g(z)| d(z) \approx 0 \text{ for internal } \mathbb{Z} \subseteq \mathbb{X} \setminus \mathbb{X}_f \]

Each \(g \in M(\mathbb{X},\mathbb{X}_f,d)\) represents a complex-valued regular Borel measure with finite variation \(\mu \in M(\mathbb{X})\)

\[ \|\mu\| \approx \|g\|_{1,d} < \infty \] (variation of \(\mu\))

\[ \int f \, d\mu \approx \sum_{x \in \mathbb{X}} f(x) g(x) d(x) \] for \(f \in C_0(\mathbb{X},E,\mathbb{X}_f)\), \(f = f^\blacklozenge \in C_0(\mathbb{X})\) (Riesz rep. thm. + Radon-Nikodym thm. " \(d\mu \approx gd(x)\) ")

Every regular complex-valued Borel measure with finite variation \(\mu \in M(\mathbb{X})\) is obtained in this way from some \(g \in M(\mathbb{X},\mathbb{X}_f,d)\); \(g\) is called lifting of \(\mu\) w.r.t. \(\mathbb{X}_f, E, d\).
\[ \mathcal{M}(X, X_f, d) \] consist of all internal functions \( g : X \to ^*\mathbb{C} \) s.t.

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Nonstandard Analysis

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Nonstandard Analysis 14

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Any group triplet $(G,G_0,G_f)$, where
• $G$ is an internal ambient group
• $G_f \leq G$ is a $\Sigma^0_1$-subgroup of finite elements of $G$
• $G_0 \triangleleft G_f$ is a $\Pi^0_1$-subgroup of infinitesimal elements of $G$
gives rise to a “topological triplet” $(G,E_G G_0,G_f)$, where
• $E_G G_0 = \{(x,y) \in G \times G; xy^{-1} \in G_0\}$
• and topological group $G^{\flat} = G_f/G_0$ – observable trace
• $G^{\flat} = G_f/G_0$ is locally compact iff
  • for any internal sets $A$, $B$, s.t. $G_0 \subseteq A \subseteq B \subseteq G_f$, there is a finite set $X = \{x_1,\ldots,x_k\} \subseteq B$ s.t. $B \subseteq AX = \bigcup_{i=1}^k Ax_i$
• $G^{\flat} = G_f/G_0$ is abelian iff $[G_f,G_f] \subseteq G_0$, i.e., $\forall x,y \in G_f: [x,y] = xyx^{-1}y^{-1} \in G_0$
PvK Duality & FT in HF Ambience 1

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PvK Duality & FT in HF Ambience
E. I. Gordon [1992]:
Every LCA group $G$ can be represented as observable trace $G ≅ G^♭ = G_f/G_0$ of some group triplet $(G, G_0, G_f)$ with hyperfinite abelian ambient group $G$. 
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**PvK Duality & FT in HF Ambience 2**

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Every LCA group \( G \) can be represented as observable trace \( G \cong G^\flat = G_f/G_0 \) of some group triplet \((G, G_0, G_f)\) with **hyperfinite** abelian ambient group \( G \).

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However small compact neighborhood \( U \) of 0 and however big compact set \( K \) in \( G \), there is a finite abelian \((U, K)\)-approximation \((F, j)\) of \( G \):

- \( F \) is finite abelian group
- \( j : F \rightarrow G \) is injective mapping
- \( K \subseteq j(F) + U = \bigcup_{a \in F} j(a) + U \)
- \( \forall a, b \in F : j(a), j(b), j(a + b) \in K \Rightarrow j(a) + j(b) - j(a + b) \in U \)
PvK Duality & FT in HF Ambience 3

In case of HF ambient group $G$, $G^{♭} = G_f/G_0$ is locally compact iff for any internal sets $A, B$, $G_0 \subseteq A \subseteq B \subseteq G_f \Rightarrow |B|/|A| < \infty$

Haar measure $m_G = m_d$ is obtained by pushing down Loeb measure $\lambda_d$ for $d = 1/|A|$, normalizing multiplier, where $A$ is arbitrary internal set s.t. $G_0 \subseteq A \subseteq G_f$

Can the dual group $\hat{G}^{♭} = \hat{G}_f/G_0$ be described in terms of some group triplet, canonically related to the original triplet $(G, G_0, G_f)$?
In case of HF ambient group $G$, $G^b = G_f/G_0$ is locally compact iff for any internal sets $A$, $B$, $G_0 \subseteq A \subseteq B \subseteq G_f \Rightarrow \frac{|B|}{|A|} < \infty$
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• Haar measure $m_G = m_d$ is obtained by pushing down
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PvK Duality & FT in HF Ambience 3

- In case of HF ambient group $G$, $G^b = G_f/G_0$ is locally compact iff for any internal sets $A$, $B$, $G_0 \subseteq A \subseteq B \subseteq G_f \Rightarrow \frac{|B|}{|A|} < \infty$

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Can the dual group $\hat{G} = \hat{G}^b = \hat{G}_f/\hat{G}_o$ be described in terms of some group triplet, canonically related to the original triplet $(G, G_0, G_f)$?
PvK Duality & FT in HF Ambience 4
PvK Duality & FT in HF Ambience

\[ \hat{G} = \ast \text{Hom}(G, \ast \mathbb{T}) \left( \cong G \cong \hat{G} \right) \text{ – internal dual group of } G: \]
all internal homomorphisms \( \gamma : G \to \ast \mathbb{T} \)
$\hat{G} = \ast \text{Hom}(G, \ast \mathbb{T}) \left( \cong G \cong \hat{G} \right)$ – internal dual group of $G$:

all *internal* homomorphisms $\gamma : G \to \ast \mathbb{T}$

$X^{\downarrow} = \{ \gamma \in \hat{G}; \ \forall \ x \in X : \gamma(x) \approx 1 \}$

$\Gamma^{\downarrow} = \{ x \in G; \ \forall \ \gamma \in \Gamma : \gamma(x) \approx 1 \}$

*infinitesimal annihilators* of arbitrary sets $X \subseteq G$, $\Gamma \subseteq \hat{G}$
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\textit{infinitesimal annihilators} of arbitrary sets \( X \subseteq G, \; \Gamma \subseteq \hat{G} \)

- \( G_0^\downarrow \) – all \textit{S}-continuous characters in \( \hat{G} \) (\( \Sigma_1^0 \))
\[ \hat{G} = \ast \text{Hom}(G, \ast \mathbb{T}) \left( \cong G \cong \hat{G} \right) \quad \text{— internal dual group of } G: \]

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- \( G_0\downarrow \) — all \( S \)-continuous characters in \( \hat{G} \) \((\Sigma^0_1)\)
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- \( (\hat{G}, G_f^\downarrow, G_0^\downarrow) \) – \textit{dual group triplet} of \( (G, G_0, G_f) \)
\( \hat{G} = \star \text{Hom}(G, \star \mathbb{T}) \left( \cong G \cong \hat{G} \right) \) \text{ – internal dual group of } G: all internal homomorphisms \( \gamma : G \to \star \mathbb{T} \)

\( X_{\downarrow} = \{ \gamma \in \hat{G}; \forall x \in X : \gamma(x) \approx 1 \} \)

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infinitesimal annihilators of arbitrary sets \( X \subseteq G, \Gamma \subseteq \hat{G} \)

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- \( G_{f}^{\downarrow} \) \text{ – all characters in } \hat{G}, \text{ infinitely close to } 1 \text{ on } G_{f} \ (\Pi_{1}^{0})
- \( (\hat{G}, G_{f}^{\downarrow}, G_{0}^{\downarrow}) \) \text{ – dual group triplet of } (G, G_{0}, G_{f})
\( \hat{G} \) = *\text{Hom}(G, *\mathbb{T}) \left( \cong G \cong \hat{G} \right) \) – internal dual group of \( G \): all internal homomorphisms \( \gamma : G \to *\mathbb{T} \)

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- \( (\hat{G}, G_f^\downarrow, G_0^\downarrow) \) – dual group triplet of \( (G, G_0, G_f) \)

What’s the relation between the observable trace \( \hat{G}^b = G_0^\downarrow / G_f^\downarrow \) of the dual triplet \( (\hat{G}, G_f^\downarrow, G_0^\downarrow) \) and the dual \( \hat{G} = \hat{G}^b = G_f / G_0 \) of the observable trace of the original triplet \( (G, G_0, G_f) \)?
PvK Duality & FT in HF Ambience 5
Example:
Example: \( \hat{G}^b = G_0^\perp / G_f^\perp \cong \hat{G_f}/\hat{G_0} = \hat{G}^b \)
Example: $\hat{G}^b = G_0^\perp / G_f^\perp \cong \hat{G}_f / G_0 = \hat{G}^b$

$K \in \mathbb{N} \setminus \mathbb{N}$, $N = 2K + 1$, $0 < d < \infty$, $Kd \not\approx 0$
Example: \( \hat{G}^\flat = G_0^\perp / G_f^\perp \cong \hat{G}_f / G_0 = \hat{G}^\flat \)

\( K \in \ast \mathbb{N} \setminus \mathbb{N}, \; N = 2K + 1, \; 0 < d < \infty, \; Kd \not\approx 0 \)

\( G = \{ nd; \; -K \leq n \leq K \} \) with \(+\) mod \( Nd \), \( (G, +) \cong (\mathbb{Z}_N, +) \)
Example: \( \hat{G}^b = G_0^\perp / G_f^\perp \cong \hat{G}_f / G_0 = \hat{G}^b \)

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Example: \( \hat{G}^b = G_0^\perp / G_f^\perp \cong G_f / G_0 = \hat{G}^b \)

\( K \in {}^*\mathbb{N} \setminus \mathbb{N}, N = 2K + 1, 0 < d < \infty, Kd \not\approx 0 \)

\( G = \{ nd; -K \leq n \leq K \} \) with \( + \mod Nd \), \( (G, +) \cong (\mathbb{Z}_N, +) \)

\( G_0 = G \cap \mathbb{I}^*\mathbb{R} = \{ x \in G; x \approx 0 \} \)

\( G_f = G \cap \mathbb{F}^*\mathbb{R} = \{ x \in G; |x| < \infty \} \)

\( G^b = G_f / G_0 \cong \begin{cases} 
\mathbb{Z} & \text{if } d \not\approx 0, \text{ as } G_0 = \{0\}, \\
\mathbb{T} & \text{if } d \approx 0, Kd < \infty, \text{ as } G_f = G, \\
\mathbb{R} & \text{if } d \approx 0, Kd \not\in \mathbb{F}^*\mathbb{R} 
\end{cases} \)
Example: \( \hat{G}^b = G_0^\perp / G_f^\perp \cong \hat{G}_f / G_0 = \hat{G}^b \)

\( K \in \ast \mathbb{N} \setminus \mathbb{N}, \; N = 2K + 1, \; 0 < d < \infty, \; Kd \not\approx 0 \)

\( G = \{ nd; \; -K \leq n \leq K \} \) with \( + \) mod \( Nd \), \( (G, +) \cong (\mathbb{Z}_N, +) \)

\( G_0 = G \cap I^* \mathbb{R} = \{ x \in G; \; x \approx 0 \} \)

\( G_f = G \cap F^* \mathbb{R} = \{ x \in G; \; |x| < \infty \} \)

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\( \hat{d} = (Nd)^{-1}, \)
Example: \( \hat{G}^b = G_0^\perp / G_f^\perp \cong G_f / G_0 = \hat{G}^b \)

\( K \in *\mathbb{N} \setminus \mathbb{N}, \ N = 2K + 1, \ 0 < d < \infty, \ Kd \not\approx 0 \)

\( G = \{ nd; \ -K \leq n \leq K \} \) with + mod \( Nd \), \( (G, +) \cong (\mathbb{Z}_N, +) \)

\( G_0 = G \cap \mathbb{I}^*\mathbb{R} = \{ x \in G; \ x \approx 0 \} \)

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\( \hat{d} = (Nd)^{-1}, \ 0 < \hat{d} < \infty, \ K\hat{d} \not\approx 0 \)
PvK Duality & FT in HF Ambience 5

Example:  \( \hat{G}^b = G_0^\perp / G_f^\perp \cong \hat{G}_f / G_0 = \hat{G}^b \)

\( K \in \ast \mathbb{N} \setminus \mathbb{N}, \ N = 2K + 1, \ 0 < d < \infty, \ Kd \not\approx 0 \)

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\( \hat{d} = (Nd)^{-1}, \ 0 < \hat{d} < \infty, \ K\hat{d} \not\approx 0 \)

\( \hat{G} = \{ nd\hat{d}; \ -K \leq n \leq K \} \) with + mod \( Nd \hat{d} \), \( (\hat{G}, +) \cong (\mathbb{Z}_N, +) \)
**PvK Duality & FT in HF Ambience 5**

**Example:** $\widehat{G^b} = G_0^\perp / G_f^\perp \cong \widehat{G_f / G_0} = \widehat{G^b}$

$K \in *\mathbb{N} \setminus \mathbb{N}, \ N = 2K + 1, \ 0 < d < \infty, \ Kd \not\approx 0$

$G = \{nd; \ -K \leq n \leq K\}$ with $+ \mod Nd, \ (G, +) \cong (\mathbb{Z}_N, +)$

$G_0 = G \cap \mathbb{I}^*\mathbb{R} = \{x \in G; \ x \approx 0\}$

$G_f = G \cap \mathbb{F}^*\mathbb{R} = \{x \in G; \ |x| < \infty\}$

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$\hat{d} = (Nd)^{-1}, \ 0 < \hat{d} < \infty, \ K\hat{d} \not\approx 0$

$\widehat{G} = \{n\hat{d}; \ -K \leq n \leq K\}$ with $+ \mod N\hat{d}, \ (\widehat{G}, +) \cong (\mathbb{Z}_N, +)$

$\widehat{G}_f^\perp = \widehat{G} \cap \mathbb{I}^*\mathbb{R} = \{y \in \widehat{G}; \ y \approx 0\}$

$\widehat{G}_0^\perp = \widehat{G} \cap \mathbb{F}^*\mathbb{R} = \{y \in \widehat{G}; \ |y| < \infty\}$
Example: \( \hat{G}^b = G_0^\perp / G_f^\perp \cong \hat{G}_f / G_0 = \hat{G}^b \)

\( K \in \mathbb{N} \setminus \mathbb{N} \), \( N = 2K + 1 \), \( 0 < d < \infty \), \( Kd \neq 0 \)

\( G = \{ nd; -K \leq n \leq K \} \) with \(+\) mod \( Nd \), \( (G, +) \cong (\mathbb{Z}_N, +) \)

\( G_0 = G \cap \mathbb{I}^\ast \mathbb{R} = \{ x \in G; x \approx 0 \} \)

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\( G^b = G_f / G_0 \cong \begin{cases} \mathbb{Z} & \text{if } d \neq 0, \text{ as } G_0 = \{ 0 \}, \\ \mathbb{T} & \text{if } d \approx 0, Kd < \infty, \text{ as } G_f = G, \\ \mathbb{R} & \text{if } d \approx 0, Kd \notin F^\ast \mathbb{R} \end{cases} \)

\( \hat{d} = (Nd)^{-1}, 0 < \hat{d} < \infty, \ K\hat{d} \neq 0 \)

\( \hat{G} = \{ nd\hat{d}; -K \leq n \leq K \} \) with \(+\) mod \( N\hat{d} \), \( (\hat{G}, +) \cong (\mathbb{Z}_N, +) \)

\( G_f^\perp = \hat{G} \cap \mathbb{I}^\ast \mathbb{R} = \{ y \in \hat{G}; y \approx 0 \} \)

\( G_0^\perp = \hat{G} \cap F^\ast \mathbb{R} = \{ y \in \hat{G}; |y| < \infty \} \)

\( \hat{G}^b = G_0^\perp / G_f^\perp \cong \)
**Example:**  \( \hat{G}^b = G_0^\perp/G_f^\perp \cong G_f/G_0 = \hat{G}^b \)

\( K \in \ast \mathbb{N} \setminus \mathbb{N}, \ N = 2K + 1, \ 0 < d < \infty, \ Kd \not\approx 0 \)

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\( G_0 = G \cap \mathbb{I}^*\mathbb{R} = \{ x \in G; \ x \approx 0 \} \)

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\( G_f^\perp = \hat{G} \cap \mathbb{I}^*\mathbb{R} = \{ y \in \hat{G}; \ y \approx 0 \} \)

\( G_0^\perp = \hat{G} \cap \mathbb{F}^*\mathbb{R} = \{ y \in \hat{G}; \ |y| < \infty \} \)

\( \hat{G}^b = G_0^\perp/G_f^\perp \cong \begin{cases} \mathbb{T} & \text{if } d \not\approx 0, \text{ as } G_0^\perp = \hat{G}, \\ \mathbb{Z} & \text{if } d \approx 0, \ Kd < \infty, \text{ as } G_f^\perp = \{ 0 \}, \\ \mathbb{R} & \text{if } d \approx 0, \ Kd \not\in \mathbb{F}^*\mathbb{R} \end{cases} \)
PvK Duality & FT in HF Ambience 6
Each $\gamma : G \to \ast\mathbb{T}$ in $G_0^\perp$ yields continuous character $\gamma^b : G \to \mathbb{T}$, by $\gamma^b(\circ x) = \circ \gamma(x)$, for $x \in G_f$:
Each $\gamma : G \to \mathbb{T}$ in $G_0^\perp$ yields continuous character $\gamma^b : G \to \mathbb{T}$, by $\gamma^b(\circ x) = \circ \gamma(x)$, for $x \in G_f$:

$$
\begin{array}{c}
G & \xleftarrow{\text{id}} & G_f & \xrightarrow{\circ} & G \\
\downarrow \gamma & & & & \downarrow \gamma^b \\
*\mathbb{T} & \xrightarrow{\circ} & *\mathbb{T}/\approx & \xleftarrow{\text{id}} & \mathbb{T}
\end{array}
$$
Each $\gamma : G \rightarrow \ast T$ in $G_0^\perp$ yields continuous character $\gamma^b : G \rightarrow T$, by $\gamma^b(\circ x) = \circ \gamma(x)$, for $x \in G_f$:

\[
\begin{array}{ccc}
G & \xleftarrow{\text{id}} & G_f & \xrightarrow{\circ} & G \\
\downarrow & & \downarrow & & \\
\ast T & \xrightarrow{\circ} & \ast T/\sim & \xleftarrow{\text{id}} & T
\end{array}
\]

- $\gamma \mapsto \gamma^b$ is group homomorphism $G_0^\perp \rightarrow \hat{G}$
Each $\gamma : G \rightarrow ^\ast \mathbb{T}$ in $G_0 \downarrow$ yields continuous character $\gamma^b : G \rightarrow \mathbb{T}$, by $\gamma^b(\circ x) = \circ \gamma(x)$, for $x \in G_f$:

$$G \leftarrow \id \quad G_f \quad \circ \quad \rightarrow G$$

$$\gamma \downarrow \qquad \downarrow \gamma^b$$

$$^\ast \mathbb{T} \quad \circ \quad ^\ast \mathbb{T}/\cong \quad \leftarrow \id \quad \mathbb{T}$$

- $\gamma \mapsto \gamma^b$ is group homomorphism $G_0 \downarrow \rightarrow \hat{G}$
- with image $\{\gamma^b; \gamma \in G_0 \downarrow\}$ and kernel $G_f$
Each \( \gamma : G \to \ast \mathbb{T} \) in \( G_0 \downarrow \) yields continuous character \( \gamma^b : G \to \mathbb{T} \), by \( \gamma^b(\circ x) = \circ \gamma(x) \), for \( x \in G_f \):

\[
\begin{array}{ccc}
G & \xleftarrow{\text{id}} & G_f & \xrightarrow{\circ} & G \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\ast \mathbb{T} & \xrightarrow{\circ} & \ast \mathbb{T}/\sim & \xleftarrow{\text{id}} & \mathbb{T}
\end{array}
\]

- \( \gamma \mapsto \gamma^b \) is group homomorphism \( G_0 \downarrow \to \hat{G} \)
- with image \( \{ \gamma^b ; \gamma \in G_0 \downarrow \} \) and kernel \( G_f \downarrow \)
- it induces injective homomorphism \( G_0 \downarrow / G_f \downarrow \to \hat{G} \)
Each $\gamma : G \rightarrow \ast \mathbb{T}$ in $G_0^\perp$ yields continuous character $\gamma^b : G \rightarrow \mathbb{T}$, by $\gamma^b(\circ x) = \circ \gamma(x)$, for $x \in G_f$:

$$
\begin{array}{ccc}
G & \leftarrow & G_f \\
\downarrow & & \downarrow \gamma^b \\
\ast \mathbb{T} & \circ & \ast \mathbb{T} / \approx \\
\end{array}
$$

- $\gamma \mapsto \gamma^b$ is group homomorphism $G_0^\perp \rightarrow \hat{G}$
- with image $\{ \gamma^b ; \gamma \in G_0^\perp \}$ and kernel $G_f^\perp$
- it induces injective homomorphism $G_0^\perp / G_f^\perp \rightarrow \hat{G}$
- this canonic mapping is isomorphism of $\hat{G}^b = G_0^\perp / G_f^\perp$ onto closed subgroup of $\hat{G} = \hat{G}^b = G_f / G_0$
PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to \ast\mathbb{T}$ in $G_0^\perp$ yields continuous character $\gamma^b : G \to \mathbb{T}$, by $\gamma^b(x) = \circ\gamma(x)$, for $x \in G_f$:

$$
\begin{array}{ccc}
G & \xleftarrow{\text{id}} & G_f & \xrightarrow{\circ} & G \\
\gamma & \downarrow & & \downarrow & \gamma^b \\
\ast\mathbb{T} & \xrightarrow{\circ} & \ast\mathbb{T}/\sim & \xleftarrow{\text{id}} & \mathbb{T}
\end{array}
$$

- $\gamma \mapsto \gamma^b$ is group homomorphism $G_0^\perp \to \hat{G}$
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- this canonic mapping is isomorphism of $\hat{G}^b = G_0^\perp/G_f^\perp$ onto closed subgroup of $\hat{G} = \hat{G}^b = G_f^\perp/G_0$
- non-$S$-continuous internal characters $\gamma \in \hat{G} \setminus G_0^\perp$ correspond neither to non-continuous characters of $G$, nor even to mappings $G \to \mathbb{T}$
Gordon’s Conjecture 1 (GC1):

The canonic mapping

\[ G \sim \mathbb{0} \rightarrow \hat{G} \sim \mathbb{0} \]

is isomorphism of topological groups.

• enough to show that \( \gamma \mapsto \gamma^{\flat} \) is onto, i.e., every character \( \gamma \in \hat{G} \) is of form \( \gamma = \gamma^{\flat} \) for some \( \gamma \in G \sim \mathbb{0} \).

• by PvK duality this is equivalent to \( G \sim \mathbb{0} \cap G \mathbb{f} = G \mathbb{0} \), i.e., characters in \( \{ \gamma^{\flat} ; \gamma \in G \sim \mathbb{0} \} \) separate points in \( G \) (density follows).

• I proved a bit more \([PZ, June – July 2012]\):

\( G \sim \mathbb{0} \mathbb{f} = G \mathbb{f} + G \sim \mathbb{0} \mathbb{f} = G \mathbb{f} \), i.e., the dual triplet of \( (\hat{G}, G \sim f, G \sim \mathbb{0}) \) is \( (G, G \mathbb{0}, G \mathbb{f}) \).

• methods: NSA + Harmonic An. + Additive Combinatorics (G. Freiman, B. Green, I. Ruzsa, T. Tao, V. Vu, ...):

analysis of Bohr sets and spectral sets

\[ \mathcal{S}_t(f) = \{ \gamma \in \hat{G} ; |\hat{f}(\gamma)| \geq t \|f\|_1 \} \]

(\( f \in C^\ast \mathbb{G}, t \in [0,1] \))
Gordon’s Conjecture 1 (GC1):
Gordon’s Conjecture 1 (GC1):
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Gordon’s Conjecture 1 (GC1):
The canonic mapping \( G_0 / G_f \rightarrow \hat{G}_f / G_0 \) is isomorphism of topological groups.

- enough to show that \( \gamma \mapsto \gamma^b \) is onto, i.e., every character \( \gamma \in \hat{G} \) is of form \( \gamma = \gamma^b \) for some \( \gamma \in G_0 \)
Gordon’s Conjecture 1 (GC1):
The canonic mapping $G_0^\perp / G_f^\perp \to \widehat{G_f} / G_0$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^b$ is onto, i.e., every character $\gamma \in \hat{G}$ is of form $\gamma = \gamma^b$ for some $\gamma \in G_0^\perp$
- by PvK duality this is equivalent to $G_0^\perp \cap G_f = G_0$, i.e., characters in $\{ \gamma^b ; \gamma \in G_0^\perp \}$ separate points in $G$ (density follows)
Gordon’s Conjecture 1 (GC1):
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- I proved a bit more [PZ, June–July 2012]: $G_0^\perp \perp = G_0$
Gordon’s Conjecture 1 (GC1):
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- enough to show that $\gamma \mapsto \gamma^\flat$ is onto, i.e., every character $\gamma \in \hat{G}$ is of form $\gamma = \gamma^\flat$ for some $\gamma \in G_0\perp$
- by PvK duality this is equivalent to $G_0\perp \perp \cap G_f = G_0$, i.e., characters in $\{\gamma^\flat; \gamma \in G_0\perp\}$ separate points in $G$ (density follows)
- I proved a bit more [PZ, June–July 2012]: $G_0\perp \perp = G_0$
- $G_f \perp \perp = G_f + G_0\perp \perp = G_f$, i.e.,
  the dual triplet of $(\hat{G}, G_f \perp, G_0\perp)$ is $(G, G_0, G_f)$
Gordon’s Conjecture 1 (GC1):
The canonic mapping \( G_0^\perp/G_f^\perp \to \widehat{G}_f/G_0 \) is isomorphism of topological groups.

- enough to show that \( \gamma \mapsto \gamma^b \) is onto, i.e., every character \( \gamma \in \hat{G} \) is of form \( \gamma = \gamma^b \) for some \( \gamma \in G_0^\perp \)

- by PvK duality this is equivalent to \( G_0^\perp \cap G_f = G_0 \), i.e., characters in \( \{ \gamma^b; \gamma \in G_0^\perp \} \) separate points in \( G \) (density follows)

- I proved a bit more [PZ, June–July 2012]: \( G_0^\perp = G_0 \)

- \( G_f^\perp = G_f + G_0^\perp = G_f \), i.e.,
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- methods: NSA + Harmonic An. + Additive Combinatorics
  (G. Freiman, B. Green, I. Ruzsa, T. Tao, V. Vu, ...):
  analysis of Bohr sets and spectral sets
  \( S_t(f) = \{ \gamma \in \hat{G}; |\hat{f}(\gamma)| \geq t \|f\|_1 \} \) \((f \in \mathbb{C}^G, \mathbb{C}^G, t \in [0, 1])\)
There are proper nontrivial subgroups $H \leq G$ s.t. $H \sim \mathbb{Z}$, hence $H \sim G$.

The equality $G \sim G_0$, in fact the inclusion $G \sim G_0 \subseteq G$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, \pi)$ and $q = (q_j)_{j=1}^{\infty}$ be sequence in $\mathbb{R}$.

Then there exists $n \in \mathbb{N}$, depending just on $\alpha, \beta$ and $q$, s.t.:

- by PvK duality: $\forall \alpha, \beta \forall q \forall G \forall (A_j) \exists n: ... \Rightarrow ...$
- gained uniformity: $\forall \alpha, \beta \forall q \exists n \forall G \forall (A_j) : ... \Rightarrow ...$
- no estimate for $n = n(\alpha, \beta, q)$
There are proper nontrivial subgroups $H \leq G$ s.t. $H^\downarrow = \{1_G\}$ (trivial character), hence $H^\downarrow \downarrow = G$. 
There are proper nontrivial subgroups $H \leq G$ s.t. $H^\perp = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

The equality $G_0^{\perp\perp} = G_0$, in fact the inclusion $G_0^{\perp\perp} \subseteq G_0$, is equivalent, in standard terms, to the following thm.
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There are proper nontrivial subgroups $H \leq G$ s.t. $H^\perp = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

The equality $G_0^{\perp\perp} = G_0$, in fact the inclusion $G_0^{\perp\perp} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $q = (q_j)_{j=1}^\infty$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on $\alpha$, $\beta$ and $q$, s.t.:
PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^\perp = \{1_G\}$ (trivial character), hence $H^{\perp \perp} = G$.

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If $G$ is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$, 

\[ \cdots \]
There are proper nontrivial subgroups $H \leq G$ s.t. $H^\perp = \{1_G\}$ (trivial character), hence $H^\perp \perp = G$.

The equality $G_0^\perp \perp = G_0$, in fact the inclusion $G_0^\perp \perp \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $q = (q_j)_{j=1}^\infty$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on $\alpha$, $\beta$ and $q$, s.t.:

If $G$ is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \quad \text{and} \quad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$
There are proper nontrivial subgroups $H \leq G$ s.t. $H^\perp = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

The equality $G_0^{\perp\perp} = G_0$, in fact the inclusion $G_0^{\perp\perp} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $q = (q_j)_{j=1}^\infty$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on $\alpha, \beta$ and $q$, s.t.:

If $G$ is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \quad \text{and} \quad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then $\mathcal{B}_{\alpha}(\mathcal{B}_{\beta}(A_n)) \subseteq A_0$. 

There are proper nontrivial subgroups $H \leq G$ s.t. $H^\perp = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

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PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^\perp = \{1_G\}$ (trivial character), hence $H^\perp \perp = G$.

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Surjectivity of canonic mapping

\[ \gamma \mapsto \gamma^{\flat} : G \sim 0 \rightarrow G/0 \rightarrow \hat{G} \]

is equivalent, in standard terms, to the following

stability thm.

Let \( \alpha, \varepsilon \in (0, 2\pi/3) \), \( k \geq 1 \) and \( q = (q_j)_{j=1}^{\infty} \), \( q_j \geq 1 \). There exist \( m \geq 1, n \geq k \) and \( \delta > 0 \), depending just on \( \alpha, \varepsilon, k \) and \( q \), s.t.:

If \( G \) is finite abelian group and \( 0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G \) are symmetric sets s.t., for \( 1 \leq j \leq n \), \( A_j + A_j \subseteq A_{j-1} \) and \( |A_{j-1}|/|A_j| \leq q_j \)

then for every partial \( \delta \)-homomorphism \( g : mA_0 \rightarrow T \), s.t.

\[ |\arg(g(x))| \leq \alpha \] for \( x \in A_k \), there exists genuine homomorphism \( \gamma : G \rightarrow T \) s.t., for each \( x \in A_0 \),

\[ |\frac{|\arg(\gamma(x)) - \arg(g(x))|}{|g(x)|} - 1| / |g(x)| \leq \varepsilon \]

"partial \( \delta \)-homomorphism" \( g : A \rightarrow T \) means:

\[ \forall x, y \in A : x + y \in A \Rightarrow |\arg(g(x+y)/g(x)g(y))| \leq \delta \]
Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^\perp / G_f^\perp \to \widehat{G}_f / G_0$ is equivalent, in standard terms, to the following stability thm.
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Let $\alpha, \varepsilon \in (0, 2\pi / 3), k \geq 1$ and $q = (q_j)_{j=1}^\infty$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on $\alpha, \varepsilon, k$ and $q$, s.t.:
Surjectivity of canonic mapping $\gamma \mapsto \gamma^b : G_0^\perp / G_f^\perp \to \widehat{G_f} / G_0$ is equivalent, in standard terms, to the following **stability** thm.

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$|\arg g(x)| \leq \alpha$ for $x \in A_k$, there exists genuine homomorphism $\gamma : G \to \mathbb{T}$ s.t., for each $x \in A_0$,

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“partial $\delta$-homomorphism” $g : A \to \mathbb{T}$ means:

$$\forall x, y \in A : x + y \in A \Rightarrow \left| \arg \frac{g(x + y)}{g(x) \cdot g(y)} \right| \leq \delta$$
Haar measure on $G = G_0/G_f$ is given as $m_G = m_d$ for normalizing multiplier $d$ s.t. $d|A| \in \mathbb{F}^*\mathbb{R}\setminus\mathbb{I}\setminus\mathbb{R}$ for some (each) internal $A$, $G_0 \subseteq A \subseteq G_f$.

Haar measure on $\hat{G} = \hat{G}_\sim = G_{\sim 0}/G_{\sim f}$ is given as $m_{\hat{G}} = \hat{m}_d$ for normalizing multiplier $\hat{d}$ s.t. $\hat{d}|B_\alpha(A)| \in \mathbb{F}^*\mathbb{R}\setminus\mathbb{I}\setminus\mathbb{R}$ for some (each) internal $A$, $G_0 \subseteq A \subseteq G_f$, $\alpha \in (0, 2\pi/3)$.

Can we have Plancherel identity and Fourier inversion formula, i.e., $d\hat{d}|G| = 1$, with such normalizing multipliers?

Gordon's Conjecture 2 (GC2): If $d$ is normalizing multiplier for the triplet $(G, G_0, G_f)$ then $\hat{d} = (d|G|)^{-1}$ is normalizing multiplier for the dual triplet $(\hat{G}, \hat{G}_\sim, \hat{G}_{\sim 0})$.

Equivalently, for internal $A$, $G_0 \subseteq A \subseteq G_f$, $\alpha \in (0, 2\pi/3)$, $|A||B_\alpha(A)| \in \mathbb{F}^*\mathbb{R}\setminus\mathbb{I}\setminus\mathbb{R}$.
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Haar measure on \( \hat{G} = G_0^\perp/G_f^\perp \) is given as \( m_{\hat{G}} = m_{\hat{d}} \)
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Haar measure on \( G = G_f/G_0 \) is given as \( m_G = m_d \) for normalizing multiplier \( d \) s.t. \( d \mid A \mid \in \mathbb{F}^{*} \mathbb{R} \setminus \mathbb{I}^{*} \mathbb{R} \) for some (each) internal \( A \), \( G_0 \subseteq A \subseteq G_f \).

Haar measure on \( \hat{G} = G_0 \downarrow / G_f \downarrow \) is given as \( m_{\hat{G}} = m_{\hat{d}} \) for normalizing multiplier \( \hat{d} \) s.t. \( \hat{d} \mid B_\alpha(A) \mid \in \mathbb{F}^{*} \mathbb{R} \setminus \mathbb{I}^{*} \mathbb{R} \) for some (each) internal \( A \), \( G_0 \subseteq A \subseteq G_f \), \( \alpha \in (0, 2\pi/3) \).

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**Gordon’s Conjecture 2 (GC2):**
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Haar measure on $G = G_f/G_0$ is given as $m_G = m_d$
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Gordon’s Conjecture 2 (GC2):
If $d$ is normalizing multiplier for the triplet $(G, G_0, G_f)$ then $\hat{d} = (d |G|)^{-1}$ is normalizing multiplier for the dual triplet $(\hat{G}, G_f^\perp, G_0^\perp)$. Equivalently, for internal $A$, $G_0 \subseteq A \subseteq G_f$, $\alpha \in (0, 2\pi/3)$,

$$\frac{|A| |B_\alpha(A)|}{|G|} \in F^* \mathbb{R} \setminus \mathbb{I}^* \mathbb{R}$$
Some accounts on the relation between Loeb measure and Haar measure show:

\[ \text{GC1} \implies \text{GC2} \]

I gave a more direct and clear proof of \text{GC2} by similar methods like those in \text{GC1}.
Some accounts on the relation between Loeb measure and Haar measure show: $\text{GC}_1 \Rightarrow \text{GC}_2$
Some accounts on the relation between Loeb measure and Haar measure show: $\text{GC}_1 \Rightarrow \text{GC}_2$

I gave a more direct and clear proof of $\text{GC}_2$ by similar methods like those in $\text{GC}_1$. 
Recall, for $f \in \mathcal{C}^*_G$, $1 \leq p < \infty$,

$$\|f\|_p = \|f\|_{p,d} = \left( \sum_{x \in G} |f(x)|^p \right)^{1/p}$$

$$\|f\|_\infty = \max_{x \in G} |f(x)|$$

Similarly, for $\varphi \in \mathcal{C}_b^*G$, with normalizing multiplier

$$\hat{d} = \left( \sum_{g \in G} |\varphi(g)| \right)^{-1}$$

$f \in \mathcal{C}^*_G$ is called $S_p$-continuous if

$$\|f_a - f\|_p \approx 0 \text{ for } a \in G_0,$$

where $f_a(x) = f(x + a)$ is the $a$-shift of $f$.

$\varphi \in \mathcal{C}^*_bG$ is called $S_p$-continuous if

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$S_\infty$-continuous is just $S_1$-continuous.
Recall, for $f \in \mathbb{C}^G$, $1 \leq p < \infty$, $d$ normalizing multiplier:
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Recall, for \( f \in \ast \mathbb{C}^G \), \( 1 \leq p < \infty \), \( d \) normalizing multiplier:

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Similarly, for \( \phi \in \ast \mathbb{C}^{\hat{G}} \), with normalizing multiplier
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$$f \in \ast \mathbb{C}^G$$ is called $S^p$-continuous if $\|f_a - f\|_p \approx 0$ for $a \in G_0$, where $f_a(x) = f(x+a)$ is the $a$-shift of $f$. 
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Recall, for \( f \in \mathbb{C}^G \), \( 1 \leq p < \infty \), \( d \) normalizing multiplier:

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Similarly, for \( \phi \in \mathbb{C}^\widehat{G} \), with normalizing multiplier \( \hat{d} = (d |G|)^{-1} \)

\( f \in \mathbb{C}^G \) is called \( S^p \)-continuous if \( \|f_a - f\|_p \approx 0 \) for \( a \in G_0 \), where \( f_a(x) = f(x + a) \) is the \( a \)-shift of \( f \).

\( \phi \in \mathbb{C}^\widehat{G} \) is called \( S^p \)-continuous if \( \|\phi_\gamma - \phi\|_p \approx 0 \) for \( \gamma \in G^\perp_f \), where \( \phi_\gamma(\chi) = \phi(\gamma \chi) \) is the \( \gamma \)-shift of \( \phi \).
Recall, for \( f \in \star \mathbb{C}^G \), \( 1 \leq p < \infty \), \( d \) normalizing multiplier:

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- \( S^\infty \)-continuous is just \( S \)-continuous
Recall:

\[ L^p(G,G_0,G_f) = \{ f \in \hat{\mathcal{C}}_G; f \text{ lifts some } f \in L^p(G,m) \} \]

\[ C^0(G,G_0,G_f) = \{ f \in \hat{\mathcal{C}}_G; f \text{ lifts some } f \in C^0(G) \} \]

\[ C^b(G,G_0,G_f) = \{ f \in \hat{\mathcal{C}}_G; f \text{ lifts some } f \in C^b(G) \} \]

Characterization of liftings.

Let \( f \in \hat{\mathcal{C}}_G \). Then \( f \in L^p(G,G_0,G_f) \) iff

- \( \| f \|_p < \infty \)
- \( \sum_{z \in \mathbb{Z}} |f(z)|^p \approx 0 \) for internal \( \mathbb{Z} \subseteq G \setminus G_f \)
- \( f \) is \( S_p \)-continuous
Recall:

\[
L^p(G; G_0; G_f) = \{ f \in \hat{C}_G; \text{f lifts some } f \in L^p(G; \mathbb{m}) \}
\]

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\[ \mathcal{L}^p(G, G_0, G_f) = \{ f \in \ast \mathbb{C}^G; \ f^p \in M(G, G_f, d) \ \& \ f \text{ is } S^p\text{-continuous} \} \]
Let $1 \leq p \leq 2 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \ast C(G)$, $\|f\|_p < \infty$. Then

- $f$ is $S_p$-continuous $\Rightarrow (\hat{f})^q \in M(\hat{G}, \hat{G})$.  
- $f^p \in M(G, G, d) \Rightarrow \hat{f}$ is $S_q$-continuous.
- $f \in L^p(G, G, 0, G) \Rightarrow \hat{f} \in L^q(\hat{G}, \hat{G})$.

In particular:

- $f \in L^1(G, G, 0, G) \Rightarrow \hat{f} \in C_0(\hat{G}, \hat{G})$ (HF dimensional version of Riemann-Lebesgue thm.)
- $f \in L^2(G, G, 0, G) \Leftrightarrow \hat{f} \in L^2(\hat{G}, \hat{G})$ (in some special cases proved by Albeverio-Gordon-Khrennikov [2000])
- $f \in M(G, G, d) \Rightarrow \hat{f} \in C_b(\hat{G}, \hat{G})$.  


Smoothness-and-Decay Principle.
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Smoothness-and-Decay Principle.

Let $1 \leq p \leq 2 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in {}^\ast \mathcal{C}^G$, $\|f\|_p < \infty$. Then

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Smoothness-and-Decay Principle.
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In particular:

- $f \in \mathcal{L}^1(G, G_0, G_f) \Rightarrow \hat{f} \in \mathcal{C}_0(\hat{G}, G_f^\perp, G_0^\perp)$
  (HF dimensional version of Riemann-Lebesgue thm.)
Smoothness-and-Decay Principle.
Let \( 1 \leq p \leq 2 \leq q \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( f \in \mathcal{C}^G \), \( \|f\|_p < \infty \). Then
\[
\begin{align*}
\bullet & \quad f \text{ is } S^p\text{-continuous} \Rightarrow (\hat{f})^q \in \mathcal{M}(\hat{G}, G_0^\perp, \hat{d}) \\
\bullet & \quad f^p \in \mathcal{M}(G, G_f, d) \Rightarrow \hat{f} \text{ is } S^q\text{-continuous} \\
\bullet & \quad f \in \mathcal{L}^p(G, G_0, G_f) \Rightarrow \hat{f} \in \mathcal{L}^q(\hat{G}, G_f^\perp, G_0^\perp)
\end{align*}
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\bullet & \quad f \in \mathcal{L}^2(G, G_0, G_f) \iff \hat{f} \in \mathcal{L}^2(\hat{G}, G_f^\perp, G_0^\perp) \\
& \quad \text{(in some special cases proved by Albeverio-Gordon-Khrennikov [2000])}
\end{align*}
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Smoothness-and-Decay Principle.

Let $1 \leq p \leq 2 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \mathcal{C}^G$, $\|f\|_p < \infty$. Then

- $f$ is $S^p$-continuous $\Rightarrow$ $(\widehat{f})^q \in \mathcal{M}(\widehat{G}, G_0^\perp, \hat{d})$
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Approximation of Fourier transform.

\[ G = G_f / G_0, \hat{G} = G_\sim / G_{\sim f}, \]

\( f \mapsto \hat{f} \) – discrete HF dimensional FT

Discrete HF dimensional FT approximates all the classical FTs:

- \( F: L^1(G) \to C_0(\hat{G}) \): \( f \in L^1(G) \) is lifting of \( f \in L^1(G) \) \( \Rightarrow \hat{f} \in C_0(\hat{G}) \) is lifting of \( F(f) \in C_0(\hat{G}) \)

- \( F: L^p(G) \to L^q(\hat{G}) \): \( f \in L^p(G) \) is lifting of \( f \in L^p(G) \) \( \Rightarrow \hat{f} \in L^q(\hat{G}) \) is lifting of \( F(f) \in L^q(\hat{G}) \) for \( p = q = 2 \) this settles Gordon's Conjecture 3 [PZ, October 2012]

- \( F: M(G) \to C_{bu}(\hat{G}) \): \( g \in M(G) \) is lifting of \( \mu \in M(G) \) \( \Rightarrow \hat{g} \in C_{bu}(\hat{G}) \) is lifting of \( F(\mu) \in C_{bu}(\hat{G}) \)
Approximation of Fourier transform.
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\[ \mathbf{G} = G_f / G_0, \quad \hat{\mathbf{G}} = G_0^{\perp} / G_f^{\perp}, \]

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  \( f \in L^p(G) \) is lifting of \( f \in L^p(G) \) \( \Rightarrow \)
  \( \hat{f} \in L^q(\hat{G}) \) is lifting of \( F(f) \in L^q(\hat{G}) \)

for \( p = q = 2 \) this settles **Gordon’s Conjecture 3**

[PZ, October 2012]

- \( F : M(G) \to C_{bu}(\hat{G}) : \)
Approximation of Fourier transform.

\( G = G_f / G_0, \hat{G} = G_0^\perp / G_f^\perp, \)

\( f \mapsto \hat{f} \) – discrete HF dimensional FT *\( \mathbb{C}^G \rightarrow \mathbb{C}^{\hat{G}} \)

Discrete HF dimensional FT approximates all the classical FTs:

- **\( \mathcal{F} : L^1(G) \rightarrow C_0(\hat{G}) : \)**
  \( f \in L^1(G) \) is lifting of \( f \in L^1(G) \) \( \Rightarrow \)
  \( \hat{f} \in C_0(\hat{G}) \) is lifting of \( \mathcal{F}(f) \in C_0(\hat{G}) \)

- **\( \mathcal{F} : L^p(G) \rightarrow L^q(\hat{G}) \) (1 < \( p \leq 2, \frac{1}{p} + \frac{1}{q} = 1) : \)**
  \( f \in L^p(G) \) is lifting of \( f \in L^p(G) \) \( \Rightarrow \)
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  \( f \in L^p(G) \) is lifting of \( f \in L^p(G) \) \( \Rightarrow \)
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for \( p = q = 2 \) this settles **Gordon’s Conjecture 3**

[PZ, October 2012]

- \( \mathcal{F} : M(G) \to C_{bu}(\hat{G}) : \)
  \( g \in M(G) \) is lifting of \( \mu \in M(G) \) \( \Rightarrow \)
Approximation of Fourier transform.

\( G = G_f/G_0, \ \hat{G} = G_0^\perp/G_f^\perp, \)

\( f \mapsto \hat{f} \) – discrete HF dimensional FT ★ \( \mathbb{C}^G \to \mathbb{C}^{\hat{G}} \)

Discrete HF dimensional FT approximates all the classical FTs:

- \( \mathcal{F} : L^1(G) \to C_0(\hat{G}) : \)
  \( f \in L^1(G) \) is lifting of \( f \in L^1(G) \) \( \Rightarrow \)
  \( \hat{f} \in C_0(\hat{G}) \) is lifting of \( \mathcal{F}(f) \in C_0(\hat{G}) \)

- \( \mathcal{F} : L^p(G) \to L^q(\hat{G}) \) (1 < \( p \leq 2, \ \frac{1}{p} + \frac{1}{q} = 1 \) :)
  \( f \in L^p(G) \) is lifting of \( f \in L^p(G) \) \( \Rightarrow \)
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for \( p = q = 2 \) this settles \textbf{Gordon’s Conjecture 3} [PZ, October 2012]

- \( \mathcal{F} : M(G) \to C_{bu}(\hat{G}) : \)
  \( g \in M(G) \) is lifting of \( \mu \in M(G) \) \( \Rightarrow \)
  \( \hat{g} \in C_{b}(\hat{G}) \) is lifting of \( \mathcal{F}(\mu) \in C_{bu}(\hat{G}) \)
TYFYAP

Thank you for your attention and patience.