Pontryagin-van Kampen Duality and Fourier Transform in Hyperfinite Ambience Gordon's Conjectures

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- G is totally disconnected iff each $\gamma \in \widehat{G}$ generates a relatively compact subgroup

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- $\mathrm{L}^1(G)$ is associative algebra under convolution
  $(f * g)(x) = \int f(x - t)\, g(t)\, \mathrm{d}\boldsymbol{m}_G(t)$
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Fourier Transform 1

$\boldsymbol{m}_G$, $\boldsymbol{m}_{\widehat{G}}$ denote *Haar measures* on G, $\widehat{G}$, resp.

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  $\langle f, g\rangle_G = \int f \cdot \overline{g} \,\mathrm{d}\boldsymbol{m}_G = \int \widehat{f} \cdot \overline{\widehat{g}} \,\mathrm{d}\boldsymbol{m}_{\widehat{G}} = \left\langle \widehat{f}, \widehat{g} \right\rangle_{\widehat{G}}$
  $f(x) = \int_{\widehat{G}} \widehat{f}(\gamma)\, \gamma(x) \,\mathrm{d}\boldsymbol{m}_{\widehat{G}}(\gamma)$

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Isn't there some "universal extension" of all the spaces $\mathrm{L}^p(G)$ $(1 \le p \le 2)$ and $\mathrm{M}(G)$, and a uniform scheme defining the Fourier transform on this extension, covering all the particular cases, like if G were finite?

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P. Vopěnka: *Odmítnutí Newtonova a Lebnizova pojetí infinitesimálního kalkulu matematiky 19. a 20. století – vyvolané ať již jejich neochotou či neschopností domyslet a dotvořit základní pojmy, o něž se původní pojetí tohoto kalkulu opíralo – bylo jedním z největších omylů nejen matematiky, ale evropské vědy vůbec.*

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- every consistent "not too big" system of standard formulas is satisfied by some object (*saturation*)

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- ${}^*\mathbb{N}$ – *hypernatural numbers*: $\mathbb{N} \prec {}^*\mathbb{N}$, ${}^*\mathbb{N} \setminus \mathbb{N} \neq \emptyset$
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- X_{f} and E are external sets; X_{f} is union and E is intersection of "not too many" internal sets (Σ^0_1 and Π^0_1)

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- for $\mathbf{X}$ locally compact, compact subsets of $\mathbf{X}$ are exactly $A^\flat = \{^\circ a;\ a \in A\}$ for internal $A \subseteq X_{\text{f}}$ ("pushing-down" A)

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Nonstandard Analysis 9

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Some of them represent standard objects of different nature: cosets of functions in Lebesgue L^p spaces, measures, distributions, etc.

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- $S \subseteq X$ is λ_d-*measurable* with finite measure $\lambda_d(S)$ iff
  $$\sup\{{}^\circ\nu_d(A);\ A \subseteq S,\ A \text{ is internal}\} = \inf\{{}^\circ\nu_d(B);\ S \subseteq B \subseteq X,\ B \text{ is internal}\}$$
  and both are finite

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- $E_{G_0} = \{(x, y) \in G \times G;\ xy^{-1} \in G_0\}$

and topological group $G^\flat = G_\mathrm{f}/G_0$ – observable trace of (G, G_0, G_f)

PvK Duality & FT in HF Ambience 2

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E. I. Gordon [1991]**:**
Every LCA group $\mathbf{G}$ can be represented as observable trace $\mathbf{G} \cong G^\flat = G_\mathrm{f}/G_0$ of some group triplet (G, G_0, G_f) with **hyperfinite** abelian ambient group G.

PvK Duality & FT in HF Ambience 2

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- $\forall\, a, b \in F \;:\; j(a), j(b), j(a+b) \in K \;\Rightarrow$
  $$\Rightarrow\; j(a) + j(b) - j(a+b) \in U$$

PvK Duality & FT in HF Ambience 3

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Can the dual group $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_\mathrm{f}/G_0}$ be described in terms of some group triplet, canonically related to the original triplet (G, G_0, G_f) ?

PvK Duality & FT in HF Ambience 4

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$\widehat{G} = {}^*\mathrm{Hom}(G, {}^*\mathbb{T}) \left(\cong G \cong \widehat{\widehat{G}}\right)$ – internal dual group of G:
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$X^{\dashv} = \{\gamma \in \widehat{G};\ \forall x \in X\ :\ \gamma(x) \approx 1\}$

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What's the relation between the observable trace $\widehat{G}^\flat = G_0^{\dashv}/G_{\mathrm{f}}^{\dashv}$ of the dual triplet $\left(\widehat{G}, G_{\mathrm{f}}^{\dashv}, G_0^{\dashv}\right)$ and the dual $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_{\mathrm{f}}/G_0}$ of the observable trace $\mathbf{G} = G^\flat = G_{\mathrm{f}}/G_0$ of the original triplet (G, G_0, G_{f}) ?

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Example:

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Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\wedge}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

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Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\smallsmile\!\downarrow}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_\mathrm{f}$:

$$\begin{array}{ccccc} G & \xleftarrow[\mathrm{id}]{} & G_\mathrm{f} & \xrightarrow[\circ]{} & \mathbf{G} \\ \gamma\big\downarrow & & & & \big\downarrow\gamma^\flat \\ {}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T} \end{array}$$

PvK Duality & FT in HF Ambience 6

Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\curlywedge}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$\begin{array}{ccccc} G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm{f}} & \xrightarrow[\circ]{} & \mathbf{G} \\ \gamma\big\downarrow & & & & \big\downarrow\gamma^\flat \\ {}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T} \end{array}$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\curlywedge} \to \widehat{\mathbf{G}}$

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Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\downarrow}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm f}$:

$$\begin{array}{ccccc} G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm f} & \xrightarrow[\circ]{} & \mathbf{G} \\ {\scriptstyle\gamma}\big\downarrow & & & & \big\downarrow{\scriptstyle\gamma^\flat} \\ {}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\!\approx & \xleftarrow{\mathrm{id}} & \mathbb{T} \end{array}$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\downarrow} \to \widehat{\mathbf{G}}$
- with image $\{\gamma^\flat;\ \gamma \in G_0^{\downarrow}\}$ and kernel $G_{\mathrm f}^{\downarrow}$

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Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\perp}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$
\begin{array}{ccccc}
G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm{f}} & \xrightarrow[\circ]{} & \mathbf{G} \\
\gamma \big\downarrow & & & & \big\downarrow \gamma^\flat \\
{}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T}
\end{array}
$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\perp} \to \widehat{\mathbf{G}}$
- with image $\{\gamma^\flat;\ \gamma \in G_0^{\perp}\}$ and kernel $G_{\mathrm{f}}^{\perp}$
- it induces injective homomorphism $G_0^{\perp}/G_{\mathrm{f}}^{\perp} \to \widehat{\mathbf{G}}$

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Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\perp}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$\begin{array}{ccccc}
G & \xleftarrow[\mathrm{id}]{} & G_{\mathrm{f}} & \xrightarrow[\circ]{} & \mathbf{G} \\
\gamma\downarrow & & & & \downarrow\gamma^\flat \\
{}^*\mathbb{T} & \xrightarrow{\circ} & {}^*\mathbb{T}/\approx & \xleftarrow{\mathrm{id}} & \mathbb{T}
\end{array}$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\perp} \to \widehat{\mathbf{G}}$
- with image $\{\gamma^\flat;\ \gamma \in G_0^{\perp}\}$ and kernel $G_{\mathrm{f}}^{\perp}$
- it induces injective homomorphism $G_0^{\perp}/G_{\mathrm{f}}^{\perp} \to \widehat{\mathbf{G}}$
- this canonic mapping is isomorphism of $\widehat{G}^\flat = G_0^{\perp}/G_{\mathrm{f}}^{\perp}$ onto closed subgroup of $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_{\mathrm{f}}/G_0}$

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Each $\gamma : G \to {}^*\mathbb{T}$ in $G_0^{\downharpoonleft}$ yields continuous character $\gamma^\flat : \mathbf{G} \to \mathbb{T}$, by $\gamma^\flat({}^\circ x) = {}^\circ\gamma(x)$, for $x \in G_{\mathrm{f}}$:

$$\begin{array}{ccccc} G & \xleftarrow[\mathrm{id}]{\quad} & G_{\mathrm{f}} & \xrightarrow[\circ]{\quad} & \mathbf{G} \\ \gamma\big\downarrow & & & & \big\downarrow\gamma^\flat \\ {}^*\mathbb{T} & \xrightarrow{\ \circ\ } & {}^*\mathbb{T}/\approx & \xleftarrow{\ \mathrm{id}\ } & \mathbb{T} \end{array}$$

- $\gamma \mapsto \gamma^\flat$ is group homomorphism $G_0^{\downharpoonleft} \to \widehat{\mathbf{G}}$
- with image $\{\gamma^\flat;\ \gamma \in G_0^{\downharpoonleft}\}$ and kernel $G_{\mathrm{f}}^{\downharpoonleft}$
- it induces injective homomorphism $G_0^{\downharpoonleft}/G_{\mathrm{f}}^{\downharpoonleft} \to \widehat{\mathbf{G}}$
- this canonic mapping is isomorphism of $\widehat{G}^\flat = G_0^{\downharpoonleft}/G_{\mathrm{f}}^{\downharpoonleft}$ onto closed subgroup of $\widehat{\mathbf{G}} = \widehat{G^\flat} = \widehat{G_{\mathrm{f}}/G_0}$
- non-S-continuous internal characters $\gamma \in \widehat{G} \setminus G_0^{\downharpoonleft}$ correspond neither to non-continuous characters of $\mathbf{G}$, nor even to mappings $\mathbf{G} \to \mathbb{T}$

PvK Duality & FT in HF Ambience 7

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\sim\!\!\downarrow}/G_{\mathrm{f}}^{\sim\!\!\downarrow} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

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Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\wr\!\downarrow}/G_{\mathrm{f}}^{\wr\!\downarrow} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^\flat$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^\flat$ for some $\gamma \in G_0^{\wr\!\downarrow}$

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\downharpoonleft}/G_{\mathrm{f}}^{\downharpoonleft} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^{\flat}$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^{\flat}$ for some $\gamma \in G_0^{\downharpoonleft}$
- by PvK duality this is equivalent to $G_0^{\downharpoonleft\downharpoonleft} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^{\flat};\ \gamma \in G_0^{\downharpoonleft}\}$ separate points in $\mathbf{G}$ (density follows)

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\dashv}/G_{\mathrm{f}}^{\dashv} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^\flat$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^\flat$ for some $\gamma \in G_0^{\dashv}$
- by PvK duality this is equivalent to $G_0^{\dashv\dashv} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^\flat;\ \gamma \in G_0^{\dashv}\}$ separate points in $\mathbf{G}$ (density follows)
- I proved a bit more [PZ, June – July 2012]: $G_0^{\dashv\dashv} = G_0$

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Gordon's Conjecture 1 (GC1):

The canonic mapping $G_0^{\curlywedge}/G_{\mathrm{f}}^{\curlywedge} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^{\flat}$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^{\flat}$ for some $\gamma \in G_0^{\curlywedge}$
- by PvK duality this is equivalent to $G_0^{\curlywedge\curlywedge} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^{\flat};\ \gamma \in G_0^{\curlywedge}\}$ separate points in $\mathbf{G}$ (density follows)
- I proved a bit more [PZ, June – July 2012]: $G_0^{\curlywedge\curlywedge} = G_0$
- $G_{\mathrm{f}}^{\curlywedge\curlywedge} = G_{\mathrm{f}} + G_0^{\curlywedge\curlywedge} = G_{\mathrm{f}}$, i.e., the dual triplet of $\left(\widehat{G}, G_{\mathrm{f}}^{\curlywedge}, G_0^{\curlywedge}\right)$ is (G, G_0, G_{f})

PvK Duality & FT in HF Ambience 7

Gordon's Conjecture 1 (GC1):
The canonic mapping $G_0^{\perp}/G_{\mathrm{f}}^{\perp} \to \widehat{G_{\mathrm{f}}/G_0}$ is isomorphism of topological groups.

- enough to show that $\gamma \mapsto \gamma^\flat$ is onto, i.e., every character $\boldsymbol{\gamma} \in \widehat{\mathbf{G}}$ is of form $\boldsymbol{\gamma} = \gamma^\flat$ for some $\gamma \in G_0^{\perp}$
- by PvK duality this is equivalent to $G_0^{\perp\perp} \cap G_{\mathrm{f}} = G_0$, i.e., characters in $\{\gamma^\flat;\ \gamma \in G_0^{\perp}\}$ separate points in $\mathbf{G}$ (density follows)
- I proved a bit more [PZ, June – July 2012]: $G_0^{\perp\perp} = G_0$
- $G_{\mathrm{f}}^{\perp\perp} = G_{\mathrm{f}} + G_0^{\perp\perp} = G_{\mathrm{f}}$, i.e., the dual triplet of $\left(\widehat{G}, G_{\mathrm{f}}^{\perp}, G_0^{\perp}\right)$ is (G, G_0, G_{f})
- *methods*: NSA + Harmonic An. + Additive Combinatorics (G. Freiman, B. Green, I. Ruzsa, T. Tao, V. Vu, ...): analysis of Bohr sets and **spectral sets** $\mathcal{S}_t(f) = \left\{\gamma \in \widehat{G};\ |\widehat{f}(\gamma)| \geq t \, \|f\|_1\right\}$ $\quad (f \in \mathbb{C}^G,\ {}^*\mathbb{C}^G,\ t \in [0,1])$

PvK Duality & FT in HF Ambience 8

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\perp} = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

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There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\downarrow} = \{1_G\}$ (trivial character), hence $H^{\downarrow\downarrow} = G$.

The equality $G_0^{\downarrow\downarrow} = G_0$, in fact the inclusion $G_0^{\downarrow\downarrow} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\dashv} = \{1_G\}$ (trivial character), hence $H^{\dashv\dashv} = G$.

The equality $G_0^{\dashv\dashv} = G_0$, in fact the inclusion $G_0^{\dashv\dashv} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

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There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\downharpoonleft} = \{1_G\}$ (trivial character), hence $H^{\downharpoonleft\downharpoonleft} = G$.

The equality $G_0^{\downharpoonleft\downharpoonleft} = G_0$, in fact the inclusion $G_0^{\downharpoonleft\downharpoonleft} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \le G$ s.t. $H^{\downarrow\!\!\!\!\smile} = \{1_G\}$ (trivial character), hence $H^{\downarrow\!\!\!\!\smile\downarrow\!\!\!\!\smile} = G$.

The equality $G_0^{\downarrow\!\!\!\!\smile\downarrow\!\!\!\!\smile} = G_0$, in fact the inclusion $G_0^{\downarrow\!\!\!\!\smile\downarrow\!\!\!\!\smile} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \ge 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \le j \le n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \le q_j$$

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\perp\!\!\!\!\perp} = \{1_G\}$ (trivial character), hence $H^{\perp\!\!\!\!\perp\perp\!\!\!\!\perp} = G$.

The equality $G_0^{\perp\!\!\!\!\perp\perp\!\!\!\!\perp} = G_0$, in fact the inclusion $G_0^{\perp\!\!\!\!\perp\perp\!\!\!\!\perp} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

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There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\curlywedge} = \{1_G\}$ (trivial character), hence $H^{\curlywedge\curlywedge} = G$.

The equality $G_0^{\curlywedge\curlywedge} = G_0$, in fact the inclusion $G_0^{\curlywedge\curlywedge} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

- by PvK duality: $\forall\, \alpha, \beta\; \forall\, \mathbf{q}\; \forall\, G\; \forall\, (A_j)\; \exists\, n\; :\; \ldots \Rightarrow \ldots$

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There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\perp} = \{1_G\}$ (trivial character), hence $H^{\perp\perp} = G$.

The equality $G_0^{\perp\perp} = G_0$, in fact the inclusion $G_0^{\perp\perp} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

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then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

- by PvK duality: $\forall\, \alpha, \beta\ \forall\, \mathbf{q}\ \forall\, G\ \forall\, (A_j)\ \exists\, n\ :\ \ldots \Rightarrow \ldots$
- gained uniformity: $\forall\, \alpha, \beta\ \forall\, \mathbf{q}\ \exists\, n\ \forall\, G\ \forall\, (A_j)\ :\ \ldots \Rightarrow \ldots$

PvK Duality & FT in HF Ambience 8

There are proper nontrivial subgroups $H \leq G$ s.t. $H^{\curlywedge} = \{1_G\}$ (trivial character), hence $H^{\curlywedge\curlywedge} = G$.

The equality $G_0^{\curlywedge\curlywedge} = G_0$, in fact the inclusion $G_0^{\curlywedge\curlywedge} \subseteq G_0$, is equivalent, in standard terms, to the following thm.

Let $\alpha, \beta \in (0, 2\pi/3)$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$ be sequence in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$, depending just on α, β and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then $\mathcal{B}_\alpha\big(\mathcal{B}_\beta(A_n)\big) \subseteq A_0$.

- by PvK duality: $\forall\, \alpha, \beta\ \forall\, \mathbf{q}\ \forall\, G\ \forall\, (A_j)\ \exists\, n\ :\ \ldots \Rightarrow \ldots$
- gained uniformity: $\forall\, \alpha, \beta\ \forall\, \mathbf{q}\ \exists\, n\ \forall\, G\ \forall\, (A_j)\ :\ \ldots \Rightarrow \ldots$
- no estimate for $n = n(\alpha, \beta, \mathbf{q})$

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PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^\perp / G_\mathrm{f}^\perp \to \widehat{G_\mathrm{f}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

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Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^{\perp}/G_{\mathrm{f}}^{\perp} \to \widehat{G_{\mathrm{f}}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^{\infty}$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^\perp / G_\mathrm{f}^\perp \to \widehat{G_\mathrm{f}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^\infty$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^\perp / G_f^\perp \to \widehat{G_f / G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^\infty$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^{\perp\!\!\!\!\perp}/G_\mathrm{f}^{\perp\!\!\!\!\perp} \to \widehat{G_\mathrm{f}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^\infty$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then for every partial δ-homomorphism $g : mA_0 \to \mathbb{T}$, s.t. $|\arg g(x)| \leq \alpha$ for $x \in A_k$, there exists genuine homomorphism $\gamma : G \to \mathbb{T}$ s.t., for each $x \in A_0$,

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^\perp / G_\mathrm{f}^\perp \to \widehat{G_\mathrm{f}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^\infty$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

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then for every partial δ-homomorphism $g : mA_0 \to \mathbb{T}$, s.t. $|\arg g(x)| \leq \alpha$ for $x \in A_k$, there exists genuine homomorphism $\gamma : G \to \mathbb{T}$ s.t., for each $x \in A_0$,

$$\left| \arg \frac{\gamma(x)}{g(x)} \right| \leq \varepsilon \, .$$

PvK Duality & FT in HF Ambience 9

Surjectivity of canonic mapping $\gamma \mapsto \gamma^\flat : G_0^\perp / G_\mathrm{f}^\perp \to \widehat{G_\mathrm{f}/G_0}$ is equivalent, in standard terms, to the following **stability** thm.

Let $\alpha, \varepsilon \in (0, 2\pi/3)$, $k \geq 1$ and $\mathbf{q} = (q_j)_{j=1}^\infty$, $q_j \geq 1$. There exist $m \geq 1$, $n \geq k$ and $\delta > 0$, depending just on α, ε, k and $\mathbf{q}$, s.t.:

If G is finite abelian group and $0 \in A_n \subseteq \ldots \subseteq A_1 \subseteq A_0 \subseteq G$ are symmetric sets s.t., for $1 \leq j \leq n$,

$$A_j + A_j \subseteq A_{j-1} \qquad \text{and} \qquad \frac{|A_{j-1}|}{|A_j|} \leq q_j$$

then for every partial δ-homomorphism $g : mA_0 \to \mathbb{T}$, s.t. $|\arg g(x)| \leq \alpha$ for $x \in A_k$, there exists genuine homomorphism $\gamma : G \to \mathbb{T}$ s.t., for each $x \in A_0$,

$$\left| \arg \frac{\gamma(x)}{g(x)} \right| \leq \varepsilon \,.$$

"partial δ-homomorphism" $g : A \to \mathbb{T}$ means:

$$\forall\, x, y \in A \,:\; x + y \in A \;\Rightarrow\; \left|\arg\big(g(x+y)/g(x)\,g(y)\big)\right| \leq \delta$$

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Haar measure on $\mathbf{G} = G_{\mathrm{f}}/G_0$ is given as $\boldsymbol{m}_G = \boldsymbol{m}_d$ for normalizing multiplier d s.t. $d\,|A| \in \mathbb{F}^*\mathbb{R} \smallsetminus \mathbb{I}^*\mathbb{R}$ for some (each) internal A, $G_0 \subseteq A \subseteq G_{\mathrm{f}}$.

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Haar measure on $\mathbf{G} = G_{\mathrm{f}}/G_0$ is given as $\boldsymbol{m}_G = \boldsymbol{m}_d$
for normalizing multiplier d s.t. $d\,|A| \in \mathbb{F}^*\mathbb{R} \setminus \mathbb{I}^*\mathbb{R}$
for some (each) internal A, $G_0 \subseteq A \subseteq G_{\mathrm{f}}$.

Haar measure on $\widehat{\mathbf{G}} = G_0^{\dashv}/G_{\mathrm{f}}^{\dashv}$ is given as $\boldsymbol{m}_{\widehat{G}} = \boldsymbol{m}_{\hat{d}}$
for normalizing multiplier $\hat{d}$ s.t. $\hat{d}\,|\mathcal{B}_\alpha(A)| \in \mathbb{F}^*\mathbb{R} \setminus \mathbb{I}^*\mathbb{R}$
for some (each) internal A, $G_0 \subseteq A \subseteq G_{\mathrm{f}}$, $\alpha \in (0, 2\pi/3)$.

PvK Duality & FT in HF Ambience 10

Haar measure on $\mathbf{G} = G_{\mathrm{f}}/G_0$ is given as $\boldsymbol{m}_G = \boldsymbol{m}_d$
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I gave a more direct and clear proof of **GC2** by similar methods like those in **GC1**.

PvK Duality & FT in HF Ambience 12

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- S^∞-continuous is just S-continuous

PvK Duality & FT in HF Ambience 13

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PvK Duality & FT in HF Ambience 13

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PvK Duality & FT in HF Ambience 14

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Liftings of complex measures. [M. Ziman & PZ 2006]

PvK Duality & FT in HF Ambience 14

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PvK Duality & FT in HF Ambience 14

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PvK Duality & FT in HF Ambience 14

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PvK Duality & FT in HF Ambience 14

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- $\int \mathbf{f}\,\mathrm{d}\mu \approx d \sum_{x \in G} f(x)\, g(x)$
  for $f \in \mathcal{C}_0(G, G_0, G_\mathrm{f})$, $\mathbf{f} = f^\flat \in \mathrm{C}_0(\mathbf{G})$
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Every regular complex-valued Borel measure with finite variation $\mu \in \mathrm{M}(\mathbf{G})$ is obtained in this way from some $g \in \mathcal{M}(G, G_0, G_\mathrm{f})$; g is called **lifting** of μ.

PvK Duality & FT in HF Ambience 15

$$\mathcal{L}^p(G, G_0, G_\mathrm{f}) = \{f \in {}^*\mathbb{C}^G;\ f^p \in \mathcal{M}(G, G_0, G_\mathrm{f}) \ \& \ f \text{ is } S^p\text{-continuous}\}$$

- the linear mapping $\mathcal{M}(G, G_0, G_\mathrm{f}) \to \mathrm{M}(\mathbf{G})$ extends the linear map $\mathcal{L}^1(G, G_0, G_\mathrm{f}) \to \mathrm{L}^1(\mathbf{G})$

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- both preserve convolution, i.e., they are homomorphisms of linear algebras

PvK Duality & FT in HF Ambience 15

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- the linear mapping $\mathcal{M}(G, G_0, G_{\mathrm{f}}) \to \mathrm{M}(\mathbf{G})$ extends the linear map $\mathcal{L}^1(G, G_0, G_{\mathrm{f}}) \to \mathrm{L}^1(\mathbf{G})$
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Smoothness-and-Decay Principle.

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Let $1 \le p \le 2 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in {}^*\mathbb{C}^G$, $\|f\|_p < \infty$. Then

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Approximation of Fourier transform. [PZ, October 2012]

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  $\widehat{f} \in \mathcal{C}_0\big(\widehat{G}, G_{\mathrm{f}}^{\perp}, G_0^{\perp}\big)$ is lifting of $\mathcal{F}(\mathbf{f}) \in \mathrm{C}_0\big(\widehat{\mathbf{G}}\big)$
- $\mathcal{F} : \mathrm{L}^p(\mathbf{G}) \to \mathrm{L}^q\big(\widehat{\mathbf{G}}\big)$ $\left(1 < p \le 2,\ \frac{1}{p} + \frac{1}{q} = 1\right)$:
  $f \in \mathcal{L}^p(G, G_0, G_{\mathrm{f}})$ is lifting of $\mathbf{f} \in \mathrm{L}^p(\mathbf{G}) \;\Rightarrow$
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  for $p = q = 2$ this settles **Gordon's Conjecture 3**
- $\mathcal{F} : \mathrm{M}(\mathbf{G}) \to \mathrm{C}\big(\widehat{\mathbf{G}}\big)$:

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Approximation of Fourier transform. [PZ, October 2012]

$\mathbf{G} = G_\mathrm{f}/G_0$, $\widehat{\mathbf{G}} = G_0^{\perp}/G_\mathrm{f}^{\perp}$,

$f \mapsto \widehat{f}$ – discrete HF dimensional FT ${}^*\mathbb{C}^G \to {}^*\mathbb{C}^{\widehat{G}}$

Discrete HF dimensional FT approximates all the classical FTs:

- $\mathcal{F} : \mathrm{L}^1(\mathbf{G}) \to \mathrm{C}_0(\widehat{\mathbf{G}})$:
  $f \in \mathcal{L}^1(G, G_0, G_\mathrm{f})$ is lifting of $\mathbf{f} \in \mathrm{L}^1(\mathbf{G}) \Rightarrow$
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- $\mathcal{F} : \mathrm{L}^p(\mathbf{G}) \to \mathrm{L}^q(\widehat{\mathbf{G}})$ $(1 < p \le 2,\ \frac{1}{p} + \frac{1}{q} = 1)$:
  $f \in \mathcal{L}^p(G, G_0, G_\mathrm{f})$ is lifting of $\mathbf{f} \in \mathrm{L}^p(\mathbf{G}) \Rightarrow$
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- $\mathcal{F} : \mathrm{M}(\mathbf{G}) \to \mathrm{C}(\widehat{\mathbf{G}})$:
  $g \in \mathcal{M}(G, G_0, G_\mathrm{f})$ is lifting of $\mu \in \mathrm{M}(\mathbf{G}) \Rightarrow$

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  $\widehat{g} \in \mathcal{C}(\widehat{G}, G_{\mathrm{f}}^{\perp}, G_0^{\perp})$ is lifting of $\mathcal{F}(\mu) \in \mathrm{C}(\widehat{\mathbf{G}})$

TYFYAP

Thank you for your attention and patience.